

Lecture Notes on Multi-loop Integral Reduction and Applied Algebraic Geometry

Yang Zhang¹

¹Department of physics, ETH Zürich, Wolfgang-Pauli-Strasse 27, 8093 Zürich, Switzerland,
yang.zhang@phys.ethz.ch.

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Chapter 1

Introduction

From childhood, we know that integral calculus is more difficult than differential calculus, moreover a multiple integral can be a hard nut to crack. Multifold integrals appear ubiquitously in science and technology, for example, to understand high order quantum interactions, we have to deal with *multi-loop Feynman integrals*. For precision LHC physics, next-to-next-to-leading order (NNLO) and even next-to-next-to-leading order (N3LO) contribution should be calculated, to be compared with the experimental data. This implies we have to compute two-loop or three-loop, non-supersymmetric, frequently massive Feynman integrals. It is a tough task.

Recall that in college, when we get a complicated integral, usually we do not compute it directly by brute force. Instead, we may first:

- Reduce the integrand. For example, given a univariate rational function integral, we use partial fraction to split the integrand into a sum of fractions, each of which contains only one pole.
- Convert the integral to residue computations. For an analytic univariate integrand, sometimes we can deform the contour of integral and make it a residue computation. The latter is often much easier than the original integral.
- Rewrite the integral by integration-by-parts (IBP).

All these basic techniques are used every day in high energy physics, and in all other branches of physics. For example, Ossola, Papadopoulos and Pittau (OPP) [OPP07, OPP08] developed a systematic one-loop integrand reduction method, in the fashion of partial fraction. This method reduces one-loop Feynman integrals to one-loop master integrals, whose coefficients can be automatically extracted from tree diagrams by unitarity analysis. Nowadays, OPP method becomes a standard programmable algorithm for computing next-to-leading order (NLO) contributions.

However, for two-loop and higher-loop Feynman integrals, these basic techniques for simplifying integrals often become insufficient. For example,

- For multi-loop orders, a Feynman integrand is still a rational function, however, in *multiple variables*. In this case, it is not easy to carry out partial fraction or general integrand reduction. This new issue is the *monomial order*, the order of variables. Naive reduction results may be too complicated for next steps, integral computation or unitarity analysis.

- For multi-loop generalized unitarity, sometimes we have residues not from one complex variable, but from multiple complex variables. It is well-known that the analysis of several complex variables is much harder than univariate complex analysis. For example,

(Hartog) Let $f(z_1, \dots, z_n)$ be an analytic function in $U \setminus \{P\}$, where U is an open set of \mathbb{C}^n ($n > 1$) and P is a point in U . Then $f(z_1, \dots, z_n)$ is analytic in U .

Hartog's theorem implies that any isolated singular point of a multivariate analytic function is removable. Hence, non-trivial singular points of multivariate analytic function have a much more complicated geometric structure than those in univariate cases. Besides, multivariate Cauchy's theorem does not apply for the case when analytic functions have zero Jacobian at the pole. That makes residue computation difficult. For instance,

$$\oint \oint_{\text{around } (0,0)} \frac{dz_1 dz_2}{(az_1^3 + z_1^2 + z_2^2)(z_1^3 + z_1 z_2 - z_2^2)} = ? \quad (1.1)$$

- For multi-loop integrals, the number of IBP relations becomes huge. We may need to list a large set of IBP relations, and then use linear algebra to eliminate unwanted terms to get *useful* IBPs. However, the linear system can be very large and Gauss elimination (especially in analytic computations) may exhaust computer RAM.

Is there a way to list only useful IBPs, by adding constraints on differential forms? The answer is “yes”, but these constraints are subtle. These are linear equations which only allow polynomial solutions [GKK11]. ¹ How do we solve them efficiently?

Most Feynman integral simplification procedures in multi-loop orders, suffer from the complicated structure of multiple variables. Note that, usually our targets are just polynomials or rational functions. However, multivariate polynomial problems can be extremely difficult. (One famous example is Jacobian conjecture, which stands unsolved today.)

The modern branch of mathematics dealing with multivariate polynomials and rational functions is *algebraic geometry*. Classically, algebraic geometry studies the geometric sets defined by zeros of polynomials. Polynomial problems are translated to geometry problems, and vice versa. Note that since only polynomials are allowed, algebraic geometry is more “rigid” than *differential geometry*. Classical algebraic geometry culminates at the classification theorem of algebraic surfaces by the *Italian school* in 19th century.

Modern algebraic geometry is rigorous, much more general and abstract. The classical geometric objects are replaced by the abstract concept *scheme*, and powerful techniques like *homological algebra* and *cohomology* are introduced in algebraic geometry thanks to Alexander Grothendieck and contemporary mathematicians [GD71, Gro61, Gro63, Gro64, Gro65, Gro66, Gro67, Har77]. Modern algebraic geometry shows its power in the proof

¹As an analogy, consider the equation $6x + 9y = 15$ in x, y . If x, y are allowed to be rational numbers, it is a simple linear equation. However, if only integer values for x, y are allowed, it is a less-trivial Diophantine equation in number theory. Here we have polynomial-valued Diophantine equations.

of *Fermat's last theorem* by Andrew Wiles. Now algebraic geometry applies on number theory, representation theory, complex geometry and theoretical physics.

Back to our cases, there are numerous polynomial/rational function problems. Clearly, they are not as sophisticated as *Fermat's last theorem* or *Riemann hypothesis*. Apparently they resemble classical algebraic geometry problems. However, beyond the classification of curves or surfaces, we need computational power to solve polynomial-form equations, to compute multivariate residues in the real world. The computational aspect of algebraic geometry, was neglected for a long time.

When I was a graduate student, I was lucky taking a class by Professor Michael Stillman. One fascinating thing in the class was that many times after learning an important theorem, Michael turned on the computer and ran a program called "Macaulay2" [GS]. He typed in number fields, polynomials, and geometric objects in the study. Then various commands in the program can automatically generate the dimension, the genus and various maps between objects. He taught us one essential tool behind the program was the so-called *Gröbner basis*, which is the crucial concept in the new subject *computational algebraic geometry* (CAG) [CLO15, CLO98]. It was my first time hearing about CAG and soon found it useful.

CAG aims at multivariate polynomial and rational function problems in the real world. It began with *Buchberger's algorithm* in 1970s, which obtained the Gröbner basis for a *polynomial ideal*. Buchberger's algorithm for polynomials is similar to Gaussian Elimination for linear algebra: the latter finds a linear basis of a subspace while the former finds a "good" generating set for an ideal. With Gröbner basis, one can carry out multivariate polynomial division and simplify rational functions; one can eliminate variables from a polynomial system; one can apply polynomial constraints without solving them... Then CAG developed quickly and now it is so all-purpose that people use it outside mathematics, like in robotics, cryptography and game theory. I believe that CAG is crucial for the deep understanding of multi-loop scattering amplitudes.

Hence, the purpose of these lecture notes is to introduce a fast-developing research field: applied algebraic geometry in multi-loop scattering amplitudes. I would like to show CAG methods by examples,

- Multi-loop integrand reduction via Gröbner basis. This generalizes one-loop OPP integrand reduction method to all loop orders. In this section, I will introduce basic notations of polynomial ring, rudiments of algebraic geometry and the Gröbner basis method.
- Multivariate residue computation, in *generalized* unitarity analysis. A flavor of several complex variables will be provided in the section. Then I present the definition of multivariate residues and CAG based algorithms for computing multivariate residues. Finally I show that they are very useful in high-loop unitarity analysis.
- Multi-loop IBP with polynomial constraints. These constraints form a *syzygy* system, which can be solved by Gröbner basis [GKK11] techniques. We show that we can combine this with unitarity cuts and the Baikov representation [Bai96] to further improve the efficiency.

I will illustrate mathematical concepts and methods by practical examples and exercises, even beyond mathematics/physics, like the game *Sudoku*. The proof of many mathemat-

ical theorems will be skipped or just roughly sketched. I will not cover all the technical details of the research frontier from integral reduction, since I believe it is more important for readers to get the idea of basic algebraic geometry and find its applications in their own research fields.

Chapter 2

Integrand reduction and Gröbner basis

2.1 Basic physical objects

In these notes we mainly focus on scattering amplitudes in perturbative quantum field theory and (super-)gravity. To make the reduction methods general, we aim at non-supersymmetric amplitudes. These methods definitely work with supersymmetric theories, however, it is more efficient to combine them with specific shortcuts in supersymmetric theories.

Referring to an L -loop Feynman diagram, we mean a *connected* diagram with n external lines, P propagators, and L *fundamental cycles*¹. We further require that each external line is connected to some fundamental cycle. Define V as the number of vertices in this diagram, then the graph theory relation holds,

$$L = P - V + 1. \quad (2.1)$$

Note that this relation is not Euler's famous formula, since this relation holds for both planar and nonplanar graphs in graph theory, but Euler characteristic does not enter this relation. (Of course, for a planar graph, by embedding fundamental cycles into a plane as face boundaries, it becomes Euler's formula for planar graphs.)

For gauge theories, we have *color-ordered* Feynman diagrams such that the external color particles must be drawn from infinity in a given cyclic order, and the Feynman rules would differ from the unordered ones. Sometimes, with these constraints, we cannot draw a Feynman diagram on a plane without crossing lines. We call such a Feynman diagram a *nonplanar diagram in the sense of color ordering*. Note that this definition is different from *nonplanar diagram in the sense of graph theory*, since by lifting the color order constraint, a colored-ordered nonplanar diagram may be embedded into a plane without crossing lines. See an example in Fig.2.1.

Sometimes, for an L -loop diagram with $L > 1$, two fundamental cycles do not share a

¹We need some graph theory concepts here: for a graph G , a *spanning tree* T is a tree subgraph which contains all vertices of G . Given any edge e in G which is not in T , we define a *fundamental cycle* C_e as the simple cycle which consists of e and a subset of T . The number of fundamental cycles is independent of the choice of T .

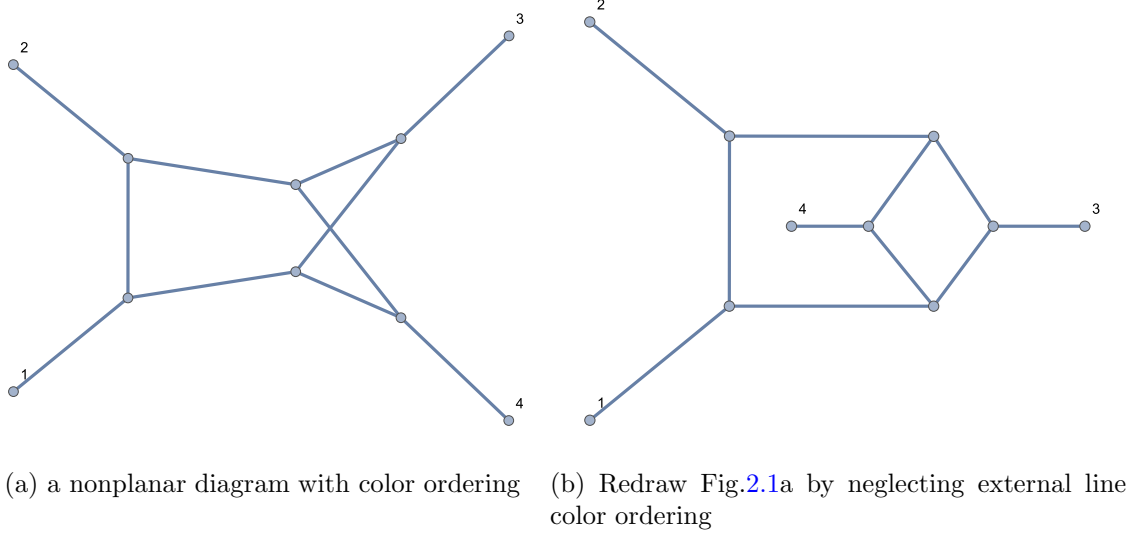


Figure 2.1: A nonplanar diagram in color ordering, may be a planar diagram in the sense of graph theory.

common edge. In this case the diagram is *factorable*, i.e., factorized into two diagram. We consider a factorable diagram as two lower loop-order diagrams, instead of an “authentic” L -loop diagram. See an example in Figure 2.2a. For a n -point L -loop diagram, if two external lines attach to one vertex, we consider this diagram as an $n - 1$ point diagram. See an example in Figure 2.2b.

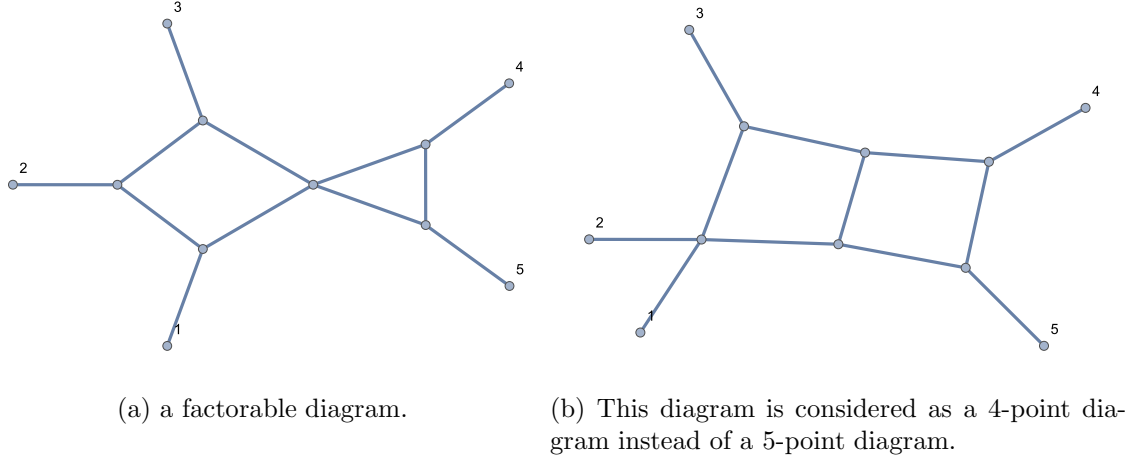


Figure 2.2: Diagrams to be simplified

A Feynman diagram has the associated Feynman integral,

$$I = \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \dots, l_L)}{D_1 \dots D_P}. \quad (2.2)$$

For each fundamental cycle, we assign an internal momenta l_i . Here the denominators of

Feynman propagator have the form, $D_i = (\alpha_1 l_1 + \dots \alpha_L l_L + \beta_1 k_1 + \dots \beta_n k_n)^2 - m_i^2$. $k_1 \dots k_n$ are the external momenta. α 's must be ± 1 . For fermion propagators, we complete the denominator squares to get this form. $N(l_1, \dots l_L)$ is the numerator, which depends on Feynman rules and the symmetry factor. Here we hide the dependence of external momenta/polarizations in $N(l_1, \dots l_L)$. The spacetime dimension D may take the value $4 - 2\epsilon$ in the *dimensional regularization scheme* (DimReg). Sometimes we also discuss the case $D = 4$ or some other fixed integer, for studying leading singularity and maximal unitarity cut.

2.2 Integrand reduction at one loop

Consider the problem of reducing the integrand in (2.2) before integration. Schematically integrand reduction, as a generalization of partial fractions, is to express the numerator N as,

$$N = \Delta + \sum_{j=1}^P h_j D_j, \quad (2.3)$$

where Δ and h_j 's are polynomials in loop momenta components. The term $h_j D_j$ cancels a denominator D_i and provides a Feynman integral with fewer propagators. Then this term merges with other Feynman integrals in the scattering amplitude. Δ remains for this diagram. If Δ is “significantly simpler” than N , this integrand reduction is useful.

2.2.1 Box diagram

To make our discussion solid, we first introduce the classical OPP reduction method [OPP07, OPP08] at one loop order. It is well known that if $D = 4$, all one-loop Feynman integrals with more than 4 distinct propagators can be reduced to Feynman integrals with at most 4 distinct propagators, while if $D = 4 - 2\epsilon$, one-loop Feynman integrals with more than 5 distinct propagators are reduced to Feynman integrals with at most 5 distinct propagators, at the integrand level. These statements can be proven by tensor calculations [Mel65]. Later in this section, we re-prove these by a straightforward algebraic geometry argument.

For simple $D = 4$ cases, we only need to start from the box diagram. For instance, consider $D = 4$ four-point massless box, with denominators in propagators,

$$D_1 = l^2, \quad D_2 = (l - k_1)^2, \quad D_3 = (l - k_1 - k_2)^2, \quad D_4 = (l + k_4)^2. \quad (2.4)$$

The Mandelstam variables are $s = (k_1 + k_2)^2$ and $t = (k_1 + k_4)^2$. It is useful to re-parameterize the loop momentum l instead of using its Lorentz components. There are several parametrization methods: (1) van Neerven-Vermaseren parameterization [vNV84] (2) spinor-helicity parameterization (3) Baikov parametrization. Here we use the straightforward van Neerven-Vermaseren parameterization, and postpone applications of other parameterizations later.

Note that by energy-momentum conservation, only external momenta k_1, k_2 and k_4 are

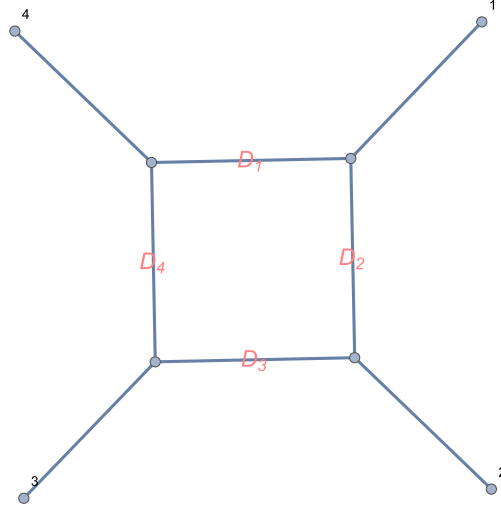


Figure 2.3: One-loop massless box diagram

independent. To make a $4D$ basis, we introduce an auxiliary vector $\omega_\mu \equiv \frac{2i}{s}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_4^\sigma$ ²

$$\omega^2 = -\frac{t(s+t)}{s}. \quad (2.5)$$

Then the basis $\{e_1, e_2 \dots e_4\} \equiv \{k_1, k_2, k_4, \omega\}$. The *Gram matrix* of this basis is,

$$G = \begin{pmatrix} 0 & \frac{s}{2} & \frac{t}{2} & 0 \\ \frac{s}{2} & 0 & \frac{-s-t}{2} & 0 \\ \frac{t}{2} & \frac{-s-t}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{t(s+t)}{s} \end{pmatrix}, \quad G_{ij} = e_i \cdot e_j. \quad (2.6)$$

Note that for any well-defined basis, Gram matrix should be non-degenerate. For any $4D$ momentum p , define van Neerven-Vermaseren variables as,

$$x_i(p) \equiv p \cdot e_i, \quad i = 1, \dots, 4. \quad (2.7)$$

Then for any two $4D$ momenta, a scalar product translates to van Neerven-Vermaseren form, by linear algebra

$$p_1 \cdot p_2 = \mathbf{x}(p_1)^T (G^{-1}) \mathbf{x}(p_2), \quad (2.8)$$

where the bold $\mathbf{x}(p)$ denotes the column 4-vector, $(x_1, x_2, x_3, x_4)^T$. Back to our one-loop box, define $x_i \equiv x_i(l)$. Hence a Lorentz-invariant numerator N_{box} in (2.2) has the form,

$$N_{\text{box}} = \sum_{m_1} \sum_{m_2} \sum_{m_3} \sum_{m_4} c_{m_1 m_2 m_3 m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}, \quad (2.9)$$

For a renormalizable theory, there is a bound on the sum, $m_1 + m_2 + m_3 + m_4 \leq 4$. The

²The normalization is from the convention of spinor helicity formalism. So ω is a pure imaginary vector, and later on the unitarity solutions appear to be real in van Neerven-Vermaseren variables.

goal in integrand reduction is to expand

$$N_{\text{box}} = \Delta_{\text{box}} + h_1 D_1 + \dots h_4 D_4, \quad (2.10)$$

such that the remainder polynomial Δ_{box} is as simple as possible.

Following [OPP07], the simplest Δ_{box} can be obtained by a direct analysis. Note that

$$\begin{aligned} x_1 &= l \cdot k_1 = \frac{1}{2}(D_1 - D_2), \\ x_2 &= l \cdot k_2 = \frac{1}{2}(D_2 - D_3) + \frac{s}{2}, \\ x_3 &= l \cdot k_4 = \frac{1}{2}(D_4 - D_1), \end{aligned} \quad (2.11)$$

hence x_1 and x_3 can be written as combinations of D_i 's, while x_2 is equivalent to the constant $s/2$ up to combinations of D_i 's. A scalar product which equals combinations of denominators and constants is called a *reducible scalar product* (RSP). In this cases, x_1, x_2, x_3 are RSPs. The remainder Δ_{box} shall not depend on RSPs, hence,

$$\Delta_{\text{box}} = \sum_{m_4} c_{m_4} x_4^{m_4}. \quad (2.12)$$

x_4 is called a *irreducible scalar product* (ISP). Furthermore, using the expansion of l^2 and (2.11),

$$\begin{aligned} D_1 = l_1^2 &= \frac{1}{4st(s+t)} \left(-4s^2x_4^2 + s^2t^2 + 4D_1s^2t - 2D_2s^2t - 2D_4s^2t + D_2^2s^2 + D_4^2s^2 \right. \\ &\quad - 2D_2D_4s^2 + 2D_1st^2 - 2D_3st^2 + 2D_1D_2st - 4D_1D_3st + 2D_2D_3st + 2D_1D_4st \\ &\quad \left. - 4D_2D_4st + 2D_3D_4st + D_1^2t^2 + D_3^2t^2 - 2D_1D_3t^2 \right), \end{aligned} \quad (2.13)$$

which means

$$x_4^2 = \frac{t^2}{4} + \mathcal{O}(D_i). \quad (2.14)$$

Hence quadratic and higher-degree monomials in x_4 should be removed from the box integrand, and

$$\Delta_{\text{box}} = c_0 + c_1(l \cdot \omega). \quad (2.15)$$

This is the *integrand basis* for the 4D box, which contains only 2 terms. Note that by Lorentz symmetry,

$$\int d^D l \frac{l \cdot \omega}{D_1 D_2 D_3 D_4} = 0, \quad (2.16)$$

for any value of D . So c_1 should not appear in the final expression of scattering amplitude. We call such a term a *spurious term*. But it is important for integrand reduction, as we will see soon.

There are two ways of using the integrand basis (2.15),

1. Direct integrand reduction (IR-D). If the numerator N is known, for instance from Feynman rules, we can use (2.11) and (2.13) to reduce N explicitly to get c_0 and

c_1 . $h_1 D_1 + \dots h_4 D_4$ is kept for further triangle, bubble ... computations.

2. Integrand reduction with unitarity (IR-U). Sometimes, it is more efficient to fit the coefficients c_0 and c_1 from tree amplitudes, by unitarity. Here c_0 and c_1 correspond to the remaining information at the quadruple cut,

$$D_1 = D_2 = D_3 = D_4 = 0. \quad (2.17)$$

From (2.11) and (2.14), there are two solutions for l , namely $l^{(1)}$ and $l^{(2)}$, characterized by,

$$(1) \quad x_1 = 0, \quad x_2 = \frac{s}{2}, \quad x_3 = 0, \quad x_4 = \frac{t}{2}, \quad (2.18)$$

$$(2) \quad x_1 = 0, \quad x_2 = \frac{s}{2}, \quad x_3 = 0, \quad x_4 = -\frac{t}{2}. \quad (2.19)$$

On this cut, the box diagram becomes four tree diagrams, summed over different *on-shell* massless internal states.

$$\begin{aligned} S_{\text{box}}^{(i)} = & \sum_{h_1} \sum_{h_2} \sum_{h_3} \sum_{h_4} A(k_1, l^{(i)} - k_1, -l^{(i)}; s_1, h_2, -h_1) \times \\ & A(k_2, l^{(i)} - k_1 - k_2, k_1 - l^{(i)}; s_2, h_3, -h_2) A(k_3, l^{(i)} + k_4, k_1 + k_2 - l^{(i)}; s_3, h_4, -h_3) \\ & \times A(k_4, l^{(i)}, -k_4 - l^{(i)}; s_4, h_1, -h_4), \end{aligned} \quad (2.20)$$

where s_i 's stand for external particles helicities, while h_i 's stand for internal particles helicities and should be summed. Unitarity implies that,

$$\begin{cases} c_0 + \frac{t}{2}c_1 = S_{\text{box}}^{(1)} \\ c_0 - \frac{t}{2}c_1 = S_{\text{box}}^{(2)} \end{cases}. \quad (2.21)$$

Generically, there is a unique solution for (c_0, c_1) . Here we see the importance of the box integrand basis (2.15). If there are fewer than 2 terms in the basis (oversimplified), then the integrand cannot be fitted from unitarity. If there are more than 2 terms in the basis (redundant), then the integrand will contain free parameters which mess up the amplitude computation for following steps.

2.2.2 Triangle diagram

After the box integrand reduction is done, we proceed to the triangle cases. Note that there are more than one triangle diagrams, in a 4-point scattering process, by pinching one internal line. Consider this one,

$$I = \int \frac{d^4 l}{i\pi^2} \frac{N_{\text{tri}}}{D_1 D_2 D_3}, \quad (2.22)$$

where external lines 3 and 4 are combined. The kinematics is much simpler than that of

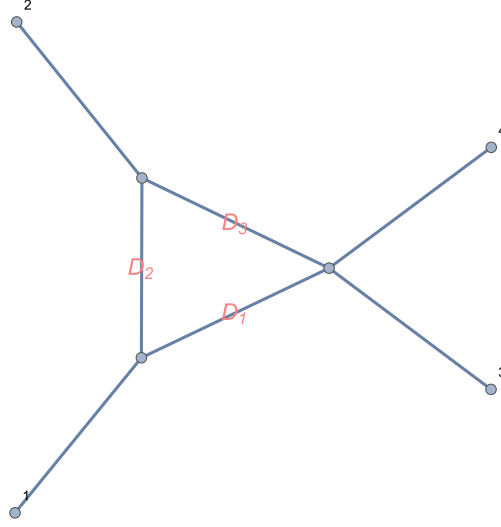


Figure 2.4: One-loop triangle diagram

the box case. Besides ω , we introduce another imaginary auxiliary vector,

$$\tilde{\omega} = i \left(-\frac{s+t}{t} k_1 + \frac{t}{s} k_2 - k_4 \right). \quad (2.23)$$

Then,

$$\tilde{\omega} \cdot k_1 = 0, \quad \tilde{\omega} \cdot k_2 = 0, \quad \omega \cdot \tilde{\omega} = 0, \quad (\tilde{\omega})^2 = \omega^2 = -\frac{t(s+t)}{s}. \quad (2.24)$$

Note that the momentum k_4 does not appear in propagators of this triangle diagram, so we would better replace the variable $x_3 = l \cdot k_4$ by a new variable $y_3 \equiv l \cdot \tilde{\omega}$,

$$x_3 = -\frac{s+t}{s} x_1 + \frac{t}{s} x_2 + i y_3. \quad (2.25)$$

The integrand reduction for triangle reads $N_{\text{tri}} = \Delta_{\text{tri}} + h_1 D_1 + h_2 D_2 + h_3 D_3$. Generically,

$$N_{\text{tri}} = \sum_{m_1} \sum_{m_2} \sum_{m_3} \sum_{m_4} d_{m_1 m_2 m_3 m_4} x_1^{m_1} x_2^{m_2} y_3^{m_3} x_4^{m_4}, \quad (2.26)$$

with the renormalization constraint that $m_1 + m_2 + m_3 + m_4 \leq 3$ [OPP07]. Here we already replaced x_3 . Again,

$$\begin{aligned} x_1 &= l \cdot p_1 = \frac{1}{2} (D_1 - D_2), \\ x_2 &= l \cdot p_2 = \frac{1}{2} (D_2 - D_3) + \frac{s}{2}. \end{aligned} \quad (2.27)$$

we have 2 RSPs, x_1, x_2 and 2 ISPs, y_3, x_4 . Again, from $D_1 = l^2$, we have

$$y_3^2 + x_4^2 = \mathcal{O}(D_i), \quad (2.28)$$

which means we can trade y_3^2 for x_4^2 . Hence with the renormalization condition,

$$\Delta_{\text{tri}} = d'_0 + d'_1 y_3 + d'_2 x_4 + d'_3 y_3 x_4 + d'_4 x_4^2 + d'_5 y_3 x_4^2 + d'_6 x_4^3. \quad (2.29)$$

which contains 7 terms. By Lorentz symmetry,

$$\int d^D l \frac{y_3^m x_4^n}{D_1 D_2 D_3} = 0, \quad (2.30)$$

as long as m is odd or n is odd. It seems that x_4^2 term survives the integration. To further simplify the integral, we redefine the integrand basis,

$$\Delta_{\text{tri}} = d_0 + d_1 y_3 + d_2 x_4 + d_3 y_3 x_4 + d_4 (x_4^2 - y_3^2) + d_5 y_3 x_4^2 + d_6 x_4^3. \quad (2.31)$$

By the symmetry between $\tilde{\omega}$ and ω , the term proportional to d_4 integrates to zero. Hence, the integrand basis of triangle contains 1 scalar integral and 6 spurious terms.³

To use this basis, again, there are two manners as in the previous section.

1. (IR-D). Suppose that the box integrand reduction is finished and the triangle diagram integrand is obtained, say from Feynman rules. We combine the triangle integrand and the term proportional to D_4 in (2.10), and carry out the reduction process in this section explicitly. Finally, we get coefficients d_0, \dots, d_6 .
2. (IR-U). The goal is to determine d_0, \dots, d_6 from unitarity. We need the triple cut,

$$D_0 = D_1 = D_3 = 0, \quad (2.32)$$

There are two branches of solutions,

$$(1) \quad x_1 = 0, \quad x_2 = \frac{s}{2}, \quad y_3 = iz, \quad x_4 = z, \quad (2.33)$$

$$(2) \quad x_1 = 0, \quad x_2 = \frac{s}{2}, \quad y_3 = -iz, \quad x_4 = z, \quad (2.34)$$

where for each branch z is a free parameter. On this cut, the numerator becomes a sum of products of tree amplitudes,

$$\begin{aligned} S_{\text{tri}}^{(i)}(z) &= \sum_{h_1} \sum_{h_2} \sum_{h_3} A(k_1, l^{(i)} - k_1, -l^{(i)}; s_1, h_2, -h_1)(z) \times \\ &A(k_2, l^{(i)} - k_1 - k_2, k_1 - l^{(i)}; s_2, h_3, -h_2)(z) A(k_3, k_4, l^{(i)}, k_1 + k_2 - l^{(i)}; s_3, s_4, h_1, -h_3)(z). \end{aligned} \quad (2.35)$$

for $i = 1, 2$. We try to fit coefficients in Δ_{tri} with $S_{\text{tri}}^{(i)}(z)$. However, the new issue is that

³We use the massless case as an illustrative example. Actually for a triangle diagram with two massless external lines, the scalar integral itself can be further reduced to bubble integrals, via IBPs.

Δ_{tri} on either branch, is a polynomial of z . $S_{\text{tri}}^{(i)}(z)$ in general is not a polynomial of z , since the last tree amplitude may have a pole when $(l + p_4)^2 = 0$. On the cut,

$$\frac{1}{(l + p_4)^2} = \frac{1}{t + 2iy_3}, \quad (2.36)$$

which becomes a fraction in z for each branch. Note that this pole is from quadruple cut, hence we have to subtract the box integrand basis to avoid the double counting. The correct unitarity relation is,

$$\Delta_{\text{tri}}(l^{(i)}(z)) = S_{\text{tri}}^{(i)}(z) - \frac{c_0 + c_1(l^{(i)}(z) \cdot \omega)}{(l^{(i)}(z) + p_4)^2}, \quad i = 1, 2. \quad (2.37)$$

If c_0 and c_1 are known from box integrand reduction, then both sides of the equation are polynomials in z and Tylor expansions determine coefficients d_0, \dots, d_6 .⁴

The further reduction for bubbles is similar.

2.2.3 D-dimensional one-loop integrand reduction

Dimensional regularization is a standard way for QFT renormalization. Here we briefly introduce OPP integrand reduction [OPP08, GKM08, EKMZ11] in D-dimension for one-loop diagrams.

Again, consider the four-point massless box integral in $D = 4 - 2\epsilon$,

$$I_{\text{box}}^D[N] = \int \frac{d^D l}{i\pi^{D/2}} \frac{N_{\text{box}}^D}{D_1 D_2 D_3 D_4}, \quad (2.38)$$

with the same definition of D_i 's. The loop momentum l contains two parts $l = l^{[4]} + l^\perp$, where $l^{[4]}$ is the four-dimensional part and l^\perp is the component in the extra dimension.

$$l^2 = (l^{[4]})^2 + (l^\perp)^2 = (l^{[4]})^2 - \mu_{11}. \quad (2.39)$$

Here we introduce a variable $\mu_{11} = -(l^\perp)^2$. We use the scheme such that all external particles are in $4D$, hence,

$$(l^\perp) \cdot k_i = 0, \quad i = 1, \dots, 4 \quad (2.40)$$

and similar orthogonal conditions hold between l^\perp and external polarization vectors hold. This implies l^\perp appears in the integrand only in the form of μ_{11} . $l^{[4]}$ is parameterized by the same van Neerven-Vermaseren variables x_1, \dots, x_4 , as before. Therefore,

$$N_{\text{box}}^D = \sum_{m_1} \sum_{m_2} \sum_{m_3} \sum_{m_4} \sum_m c_{m_1 m_2 m_3 m_4 m} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} \mu_{11}^m, \quad (2.41)$$

with the renormalization condition $m_1 + m_2 + m_3 + m_4 + 2m \leq 4$. (μ_{11} contains 2 powers

⁴Note that in general, for a massive triangle diagram, the two cut branches may merge into one. In this case, a Laurent expansion over z is needed and (2.37) again remove the redundant pole.

of l .) Again, as in the 4D case,

$$x_1 = \frac{1}{2}(D_1 - D_2), \quad x_2 = \frac{1}{2}(D_2 - D_3) + \frac{s}{2}, \quad x_3 = \frac{1}{2}(D_4 - D_1), \quad (2.42)$$

so x_1 , x_2 and x_3 are RSPs which do not appear in the integrand basis. The ISPs are x_4 and μ_{11} . From the relation $D_1 = (l^{[4]})^2 - \mu_{11}$, we get,

$$x_4^2 = \frac{t^2}{4} - \frac{(s+t)t}{s}\mu_{11} + \mathcal{O}(D_i), \quad (2.43)$$

Hence we can trade x_4^2 for μ_{11} in the integrand basis,

$$\Delta_{\text{box}}^D = c_0 + c_1 x_4 + c_2 \mu_{11} + c_3 \mu_{11} x_4 + c_4 \mu_{11}^2, \quad (2.44)$$

which contains 5 terms. The terms proportional to x_4 are again spurious, i.e., integrated to zero.

The coefficients c_0, \dots, c_4 can either be calculated from explicit reduction (IR-D) or unitarity (IR-U). For the latter, the quadruple cut $D_1 = D_2 = D_3 = D_4 = 0$ is applied. There is one family of solutions which is one-dimensional,

$$x_1 = 0, \quad x_2 = \frac{s}{2}, \quad x_3 = 0, \quad x_4 = z, \quad \mu_{11} = \frac{s(t^2 - 4z^2)}{4t(s+t)}. \quad (2.45)$$

Amazingly, the $4D$ quadruple cut contains two zero-dimensional solutions while D -dim quadruple cut has only one family of solution. The two roots in $4D$ are connected by a cut-solution curve, in DimReg. The Taylor series in z fits coefficients c_0, \dots, c_4 .

If only $\epsilon \rightarrow 0$ limit of the amplitudes is needed, (2.44) can be further simplified by *dimension shift* identities,

$$\int \frac{d^D}{i\pi^{D/2}} \frac{\mu_{11}}{D_1 D_2 D_3 D_4} = \frac{D-4}{2} I_{\text{box}}^{D+2}[1] \quad (2.46)$$

$$\int \frac{d^D}{i\pi^{D/2}} \frac{\mu_{11}^2}{D_1 D_2 D_3 D_4} = \frac{(D-4)(D-2)}{4} I_{\text{box}}^{D+4}[1] \quad (2.47)$$

These identities can be proven via Baikov parameterization (Chapter 4). It is well known that the $6D$ scalar box integral is finite and the $8D$ scalar box is UV divergent such that,

$$\lim_{D \rightarrow 4} \frac{D-4}{2} I_{\text{box}}^{D+2}[1] = 0, \quad (2.48)$$

$$\lim_{D \rightarrow 4} \frac{(D-4)(D-2)}{4} I_{\text{box}}^{D+4}[1] = -\frac{1}{3}. \quad (2.49)$$

Hence the integrand basis after integration becomes,

$$\lim_{D \rightarrow 4} \int \frac{d^D l}{i\pi^{D/2}} \frac{\Delta_{\text{box}}^D}{D_1 D_2 D_3 D_4} = c_0 I_{\text{box}}^D[1] - \frac{1}{3} c_4 \quad (2.50)$$

in the $\epsilon \rightarrow 0$ limit. The second term is called a *rational term*, which cannot be obtained

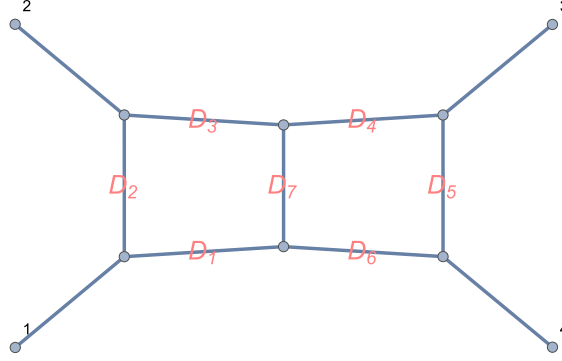


Figure 2.5: two-loop double box diagram

from the $4D$ quadruple cut.

It seems that D -dimensional integrand reduction is more complicated than the $4D$ case, with more variables and more integrals in the basis. However, it provides the complete amplitude for a general renormalizable QFT, and mathematically, its cut solution has simpler structure.

OPP method is programmable and highly efficient for automatic one-loop amplitude computation [OPP08, BBU11, CGH⁺11, HFF⁺11].

2.3 Issues at higher loop orders

Since OPP method is very convenient for one-loop cases, the natural question is: is it possible to generalize OPP method for higher loop orders?

Of course, higher loop diagrams contain more loop momenta and usually more propagators. Is it a straightforward generalization? The answer is “no”. For example, consider the $4D$ 4-point massless double box diagram (see Fig. 2.5), associated with the integral,

$$I_{\text{dbox}}[N] = \int \frac{d^4 l_1}{i\pi^2} \frac{d^4 l_2}{i\pi^2} \frac{N}{D_1 D_2 D_3 D_4 D_5 D_6 D_7}. \quad (2.51)$$

The denominators of propagators are,

$$\begin{aligned} D_1 &= l_1^2, & D_2 &= (l_1 - k_1)^2, & D_3 &= (l_1 - k_1 - k_2)^2, & D_4 &= (l_2 + k_1 + k_2)^2, \\ D_5 &= (l_2 - k_4)^2, & D_6 &= l_2^2, & D_7 &= (l_1 + l_2)^2. \end{aligned} \quad (2.52)$$

The goal of reduction is to express,

$$N_{\text{dbox}} = \Delta_{\text{dbox}} + h_1 D_1 + \dots + h_7 D_7 \quad (2.53)$$

such that Δ_{dbox} is the “simplest”. (In the sense that all its coefficients in Δ_{dbox} can be uniquely fixed from unitarity, as in the box case.)

We use van Neerven-Vermaseren basis as before, $\{e_1, e_2, e_3, e_4\} = \{k_1, k_2, k_4, \omega\}$. De-

fine

$$x_i = l_1 \cdot e_i, \quad y_i = l_2 \cdot e_i, \quad i = 1, \dots, 4. \quad (2.54)$$

Then we try to determine Δ_{dbox} in these variables like one-loop OPP method.

$$\begin{aligned} x_1 &= \frac{1}{2}(D_1 - D_2), \\ x_2 &= \frac{1}{2}(D_2 - D_3) + \frac{s}{2}, \\ y_2 &= \frac{1}{2}(D_4 - D_6) - y_1 - \frac{s}{2}, \\ y_3 &= \frac{1}{2}(D_6 - D_5), \end{aligned} \quad (2.55)$$

Hence we can remove RSPs: x_1, x_2, y_2 and y_3 in Δ_{dbox} . (We trade y_2 for y_1 , by symmetry consideration: under the left-right flip symmetry of double box, $x_3 \leftrightarrow y_1$.) There are 4 ISPs, x_3, y_1, x_4 and y_4 .

Then following the one-loop OPP approach, the quadratic terms in $(l_i \cdot \omega)$ can be removed from the integrand basis, since,

$$\begin{aligned} x_4^2 &= x_3^2 - tx_3 + \frac{t^2}{4} + \mathcal{O}(D_i), \\ y_4^2 &= y_1^2 - ty_1 + \frac{t^2}{4} + \mathcal{O}(D_i), \\ x_4 y_4 &= \frac{s+2t}{s} x_3 y_1 + \frac{t}{2} x_3 + \frac{t}{2} y_1 - \frac{t^2}{4} + \mathcal{O}(D_i). \end{aligned} \quad (2.56)$$

Then the trial version of integrand basis has the form,

$$\Delta_{\text{dbox}} = \sum_m \sum_n \sum_\alpha \sum_\beta c_{m,n,\alpha,\beta} x_3^m y_1^n x_4^\alpha y_4^\beta, \quad (2.57)$$

where $(\alpha, \beta) \in \{(0,0), (1,0), (0,1)\}$. The renormalization condition is,

$$m + \alpha \leq 4, \quad n + \beta \leq 4, \quad m + n + \alpha + \beta \leq 6. \quad (2.58)$$

By counting, there are 56 terms in the basis. Is this basis correct?

Have a look at the unitarity solution. The heptacut $D_1 = \dots D_7 = 0$ has a complicated solution structure [KL12]. (See table. 2.1). There are 6 branches of solutions, each of which is parameterized by a free parameter z_i . Solutions (5) and (6) contain poles in z_i , hence we need Laurent series for tree products,

$$S^{(i)} = \sum_{k=-4}^4 d_k^{(i)} z_i^k, \quad i = 5, 6. \quad (2.59)$$

The bounds are from renormalization conditions, so there are 9 nonzero coefficients for

	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
(1)	0	$\frac{s}{2}$	z_1	$z_1 - \frac{t}{2}$	0	$-\frac{s}{2}$	0	$\frac{t}{2}$
(2)	0	$\frac{s}{2}$	z_2	$-z_2 + \frac{t}{2}$	0	$-\frac{s}{2}$	0	$-\frac{t}{2}$
(3)	0	$\frac{s}{2}$	0	$\frac{t}{2}$	z_3	$-z_3 - \frac{s}{2}$	0	$z_3 - \frac{t}{2}$
(4)	0	$\frac{s}{2}$	0	$-\frac{t}{2}$	z_4	$-z_4 - \frac{s}{2}$	0	$-z_4 + \frac{t}{2}$
(5)	0	$\frac{s}{2}$	$\frac{z_5-s}{2}$	$\frac{z_5-s-t}{2}$	$\frac{s(s+t-z_5)}{2z_5}$	$-\frac{s(s+t)}{2z_5}$	0	$\frac{(s+t)(s-z_5)}{2z_5}$
(6)	0	$\frac{s}{2}$	$\frac{z_6-s}{2}$	$\frac{-z_6+s+t}{2}$	$\frac{s(s+t-z_6)}{2z_6}$	$-\frac{s(s+t)}{2z_6}$	0	$-\frac{(s+t)(s-z_6)}{2z_6}$

Table 2.1: solutions of the 4D double box heptacut.

each case. Solutions (1), (2), (3), (4) are relatively simpler,

$$S^{(i)} = \sum_{k=0}^4 d_k^{(i)} z_i^k, \quad i = 1, 2, 3, 4. \quad (2.60)$$

So there are 5 nonzero coefficients for each case. These solutions are not completely independent, for example, solution (1) at $z_1 = s$ and solution (6) at $z_6 = t/2$ correspond to the same loop momenta. Therefore,

$$S^{(1)}(z_1 \rightarrow s) = S^{(6)}(z_6 \rightarrow t/2). \quad (2.61)$$

There are 6 such intersections, namely between solutions (1) and (6), (1) and (4), (2) and (3), (2) and (5), (3) and (6), (4) and (5). Hence, there are $9 \times 2 + 5 \times 4 - 6 = 32$ independent $d_k^{(i)}$'s.

Now the big problem emerges,

$$56 > 32. \quad (2.62)$$

There are more terms in the integrand basis than those determined from unitarity cut. That means this integrand basis is redundant. However, it seems that we already used all algebraic constraints in (2.55) and (2.56). Which constraint is missing?

We need to reconsider (2.53), especially the meaning of “simplest” integrand basis. For simple example like massless double box diagram, it is possible to use the detailed structures like symmetries and Gram determinant constraints, to get a proper integrand basis [MO11, BFZ12b]. However, in general, we need an automatic reduction method, without looking at the details. So we refer to a new mathematical approach, *computational algebraic geometry*.

2.4 Elementary computational algebraic geometry methods

2.4.1 Basic facts of algebraic geometry in affine space I

In order to apply the new method, we need to list some basic concepts and facts on algebraic geometry [Har77].

We start from a polynomial ring $R = \mathbb{F}[z_1, \dots, z_n]$ which is the collection of all poly-

nomials in n variables z_1, \dots, z_n with coefficients in the *field* \mathbb{F} . For example, \mathbb{F} can be \mathbb{Q} , the rational numbers, \mathbb{C} , the complex numbers, $\mathbb{Z}/p\mathbb{Z}$, the *finite field* of integers modulo a prime number p , or $\mathbb{C}(c_1, c_2, \dots, c_k)$, the complex rational functions of parameters c_1, \dots, c_k .

Recall that the right hand side of (2.53) contains the sum $h_1 D_1 + \dots + h_7 D_7$ where D_i 's are known polynomials and h_i 's are arbitrary polynomials. What are general properties of such a sum? That leads to the concept of *ideal*.

Definition 2.1. *An ideal I in the polynomial ring $R = \mathbb{F}[z_1, \dots, z_n]$ is a subset of R such that,*

- $0 \in I$. For any two $f_1, f_2 \in I$, $f_1 + f_2 \in I$. For any $f \in I$, $-f \in I$.
- For $\forall f \in I$ and $\forall h \in R$, $hf \in I$.

The ideal in the polynomial ring $R = \mathbb{F}[z_1, \dots, z_n]$ generated by a subset S of R is the collection of all such polynomials,

$$\sum_i h_i f_i, \quad h_i \in R, \quad f_i \in S. \quad (2.63)$$

This ideal is denoted as $\langle S \rangle$. In particular, $\langle 1 \rangle = R$, which is an ideal which contains all polynomials. Note that even if S is an infinite set, the sum in (2.63) is always restricted to a sum of a finite number of terms. S is called the generating set of this ideal.

Example 2.2. *Let $I = \langle x^2 + y^2 + z^2 - 1, z \rangle$ in $\mathbb{Q}[x, y, z]$. By definition,*

$$I = \{h_1(x^2 + y^2 + z^2 - 1) + h_2 \cdot z, \quad \forall h_1, h_2 \in R\}, \quad (2.64)$$

Pick up $h_1 = 1$, $h_2 = -z$, and we see $x^2 + y^2 - 1 \in I$. Furthermore,

$$x^2 + y^2 + z^2 - 1 = (x^2 + y^2 - 1) + z \cdot z. \quad (2.65)$$

Hence $I = \langle x^2 + y^2 - 1, z \rangle$. We see that, in general, the generating set of an ideal is not unique.

Our integrand reduction problem can be rephrased as: given N and the ideal $I = \langle D_1, \dots, D_7 \rangle$, how many terms in N are in I ? To answer this, we need to study properties of ideals.

Theorem 2.3 (Noether). *The generating set of an ideal I of $R = \mathbb{F}[z_1, \dots, z_n]$ can always be chosen to be finite.*

Proof. See Zariski, Samuel [ZS75a]. □

This theorem implies that we only need to consider ideals generated by finite sets in the polynomial ring R .

Definition 2.4. *Let I be an ideal of R , we define an equivalence relation,*

$$f \sim g, \quad \text{if and only if } f - g \in I. \quad (2.66)$$

We define an equivalence class, $[f]$ as the set of all $g \in R$ such that $g \sim f$. The quotient ring R/I is set of equivalence classes,

$$R/I = \{[f] | f \in R\}. \quad (2.67)$$

with multiplication $[f_1][f_2] \equiv [f_1 f_2]$. (Check this multiplication is well-defined.)

To study the structure of an ideal, it is very useful to consider the algebra-geometry relation.

Definition 2.5. Let \mathbb{K} be a field, $\mathbb{F} \subset \mathbb{K}$. The n -dimensional \mathbb{K} -affine space $\mathbf{A}_{\mathbb{K}}^n$ is the set of all n -tuple of \mathbb{K} . Given a subset S of the polynomial ring $\mathbb{F}[z_1, \dots, z_n]$, its algebraic set over \mathbb{K} is,

$$\mathcal{Z}_{\mathbb{K}}(S) = \{p \in \mathbf{A}_{\mathbb{K}}^n | f(p) = 0, \text{ for every } f \in S\}. \quad (2.68)$$

If $\mathbb{K} = \mathbb{F}$, we drop the subscript \mathbb{K} in $\mathbf{A}_{\mathbb{K}}^n$ and $\mathcal{Z}_{\mathbb{K}}(S)$.

So the algebraic set $\mathcal{Z}(S)$ consists of all *common solutions* of polynomials in S . Note that to solve polynomials in S is equivalent to solve all polynomials simultaneously in the ideal generated by S ,

$$\mathcal{Z}(S) = \mathcal{Z}(\langle S \rangle), \quad (2.69)$$

since if $p \in \mathcal{Z}(S)$, then $f(p) = 0, \forall f \in S$. Hence,

$$h_1(p)f_1(p) + \dots + h_k(p)f_k(p) = 0, \quad \forall h_i \in R, \forall f_i \in S. \quad (2.70)$$

So we always consider the algebraic set of an ideal.

For example, $\mathcal{Z}(\langle 1 \rangle) = \emptyset$ (empty set) since $1 \neq 0$. For the ideal $I = \langle x^2 + y^2 + z^2 - 1, z \rangle$ in example 2.2, $\mathcal{Z}(I)$ is the unit circle on the plane $z = 0$.

We want to learn the structure of an ideal from its algebraic set. First, for the empty algebraic set,

Theorem 2.6 (Hilbert's weak Nullstellensatz). Let I be an ideal of $\mathbb{F}[z_1, \dots, z_n]$ and \mathbb{K} be an algebraically closed field⁵, $\mathbb{F} \subset \mathbb{K}$. If $\mathcal{Z}_{\mathbb{K}}(I) = \emptyset$, then $I = \langle 1 \rangle$.

Proof. See Zariski and Samuel, [ZS75b, Chapter 7]. □

Remark. The field extension \mathbb{K} must be algebraically closed. Otherwise, say, $\mathbb{K} = \mathbb{F} = \mathbb{Q}$, the ideal $\langle x^2 - 2 \rangle$ has empty algebraic set in \mathbb{Q} . (The solutions are not rational). However, $\langle x^2 - 2 \rangle \neq \langle 1 \rangle$. On the other hand, \mathbb{F} need not be algebraically closed. $I = \langle 1 \rangle$ means,

$$1 = h_1 f_1 + \dots + h_k f_k, \quad f_i \in I, \quad h_i \in \mathbb{F}[z_1, \dots, z_n]. \quad (2.71)$$

where h_i 's coefficients are in \mathbb{F} , instead of an algebraic extension of \mathbb{F} .

Example 2.7. We prove that, generally, the 4D pentagon diagrams are reduced to diagrams with fewer than 5 propagators, D -dimensional hexagon diagram are reduced to diagrams with fewer than 6 propagators, in the integrand level.

⁵A field \mathbb{K} is algebraically closed, if any non-constant polynomial in $\mathbb{K}[x]$ has a solution in \mathbb{K} . \mathbb{Q} is not algebraically closed, the set of all algebraic numbers $\bar{\mathbb{Q}}$ and \mathbb{C} are algebraically closed.

For the 4D pentagon case, there are 5 denominators from propagators, namely D_1, \dots, D_5 . There are 4 Van Neerven-Vermaseren variables for the loop momenta, namely x_1, x_2, x_3 and x_4 . So D_i 's are polynomials in x_1, \dots, x_4 with coefficients in $\mathbb{F} = \mathbb{Q}(s_{12}, s_{23}, s_{34}, s_{45}, s_{15})$. Define $I = \langle D_1, \dots, D_5 \rangle$. Generally 5 equations in 4 variables,

$$D_1 = D_2 = D_3 = D_4 = D_5 = 0, \quad (2.72)$$

have no solution (even with algebraic extensions). Hence by Hilbert's weak Nullstellensatz, $I = \langle 1 \rangle$. Explicitly, there exist 5 polynomials f_i 's in $\mathbb{F}[x_1, x_2, x_3, x_4]$ such that

$$f_1 D_1 + f_2 D_2 + f_3 D_3 + f_4 D_4 + f_5 D_5 = 1. \quad (2.73)$$

Therefore,

$$\begin{aligned} \int d^4l \frac{1}{D_1 D_2 D_3 D_4 D_5} &= \int d^4l \frac{f_1}{D_2 D_3 D_4 D_5} + \int d^4l \frac{f_2}{D_1 D_3 D_4 D_5} + \int d^4l \frac{f_3}{D_1 D_2 D_4 D_5} \\ &\quad + \int d^4l \frac{f_4}{D_1 D_2 D_3 D_5} + \int d^4l \frac{f_5}{D_1 D_2 D_3 D_4}, \end{aligned} \quad (2.74)$$

where each term in the r.h.s is a box integral (or simpler). Note that f_i 's are in $\mathbb{F}[x_1, x_2, x_3, x_4]$, so the coefficients of these polynomials are rational functions of Mandelstam variables $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$. Weak Nullstellensatz theorem does not provide an algorithm for finding such f_i 's. The algorithm will be given by the Gröbner basis method in next subsection, or by the resultant method [CLO98].

Notice that in the DimReg case, we have one more variable $\mu_{11} = -(l^\perp)^2$. The same argument using Weak Nullstellensatz leads to the result.

For a general algebraic set, we have the important theorem:

Theorem 2.8 (Hilbert's Nullstellensatz). *Let \mathbb{F} be an algebraically closed field and $R = \mathbb{F}[z_1, \dots, z_n]$. Let I be an ideal of R . If $f \in R$ and,*

$$f(p) = 0, \quad \forall p \in \mathcal{Z}(I), \quad (2.75)$$

then there exists a positive integer k such that $f^k \in I$.

Proof. See Zariski and Samuel, [ZS75b, Chapter 7]. □

Hilbert's Nullstellensatz characterizes all polynomials vanishing on $\mathcal{Z}(I)$, they are “not far away” from elements in I . For example, $I = \langle (x-1)^2 \rangle$ and $\mathcal{Z}(I) = \{1\}$. The polynomial $f(x) = (x-1)$ does not belong to I but $f^2 \in I$.

Definition 2.9. *Let I be an ideal in R , define the radical ideal of I as,*

$$\sqrt{I} = \{f \in R | \exists k \in \mathbb{Z}^+, f^k \in I\}. \quad (2.76)$$

For any subset V of \mathbf{A}^n , define the ideal of V as

$$\mathcal{I}(V) = \{f \in R | f(p) = 0, \forall p \in V\}. \quad (2.77)$$

Then Hilbert's Nullstellensatz reads, over an algebraically closed field,

$$\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}. \quad (2.78)$$

An ideal I is called radical, if $\sqrt{I} = I$.

If two ideals I_1 and I_2 have the same algebraic set $\mathcal{Z}(I_1) = \mathcal{Z}(I_2)$, then they have the same radical ideals $\sqrt{I_1} = \sqrt{I_2}$. On the other hand, if two sets in \mathbb{A}^n have the same ideal, what could we say about them? To answer this question, we need to define topology of \mathbb{A}^n :

Definition 2.10 (Zariski topology). Define Zariski topology of $\mathbb{A}_{\mathbb{F}}^n$ by setting all algebraic set to be topologically closed. (Here \mathbb{F} need not be algebraic closed.)

Remark. The intersection of any number of Zariski closed sets is closed since,

$$\bigcap_i \mathcal{Z}(I_i) = \mathcal{Z}\left(\bigcup_i I_i\right). \quad (2.79)$$

The union of two closed sets is closed since,

$$\mathcal{Z}(I_1) \cup \mathcal{Z}(I_2) = \mathcal{Z}(I_1 I_2) = \mathcal{Z}(I_1 \cap I_2). \quad (2.80)$$

$\mathbb{A}_{\mathbb{F}}^n$ and \emptyset are both closed because $\mathbb{A}_{\mathbb{F}}^n = \mathcal{Z}(\{0\})$, $\emptyset = \mathcal{Z}(\langle 1 \rangle)$. That means Zariski topology is well-defined. We leave the proof of (2.79) and (2.80) as an exercise.

Note that Zariski topology is different from the usual topology defined by Euclidean distance, for $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For example, over \mathbb{C} , the “open” unit disc defined by $D = \{z \mid |z| < 1\}$ is not Zariski open in $\mathbb{A}_{\mathbb{C}}^1$. The reason is that $\mathbb{C} - D = \{z \mid |z| \geq 1\}$ is not Zariski closed, i.e. $\mathbb{C} - D$ cannot be the solution set of one or several complex polynomials in z .

Zariski topology is the foundation of affine algebraic geometry. With this topology, the dictionary between algebra and geometry can be established.

Proposition 2.11. (Here \mathbb{F} need not be algebraic closed.)

1. If $I_1 \subset I_2$ are ideals of $\mathbb{F}[z_1, \dots, z_n]$, $\mathcal{Z}(I_1) \supset \mathcal{Z}(I_2)$
2. If $V_1 \subset V_2$ are subsets of $\mathbb{A}_{\mathbb{F}}^n$, $\mathcal{I}(V_1) \supset \mathcal{I}(V_2)$
3. For any subset V in $\mathbb{A}_{\mathbb{F}}^n$, $\mathcal{Z}(\mathcal{I}(V)) = \overline{V}$, the Zariski closure of V .

Proof. The first two statements follow directly from the definitions. For the third one, $V \subset \mathcal{Z}(\mathcal{I}(V))$. Since the latter is Zariski closed, $\overline{V} \subset \mathcal{Z}(\mathcal{I}(V))$. On the other hand, for any Zariski closed set X containing V , $X = \mathcal{Z}(I)$. $I \subset \mathcal{I}(V)$. From statement 1, $X = \mathcal{Z}(I) \supset \mathcal{Z}(\mathcal{I}(V))$. As a closed set, $\mathcal{Z}(\mathcal{I}(V))$ is contained in any closed set which contains V , hence $\mathcal{Z}(\mathcal{I}(V)) = \overline{V}$. \square

In the case \mathbb{F} is algebraic closed, the above proposition and Hilbert's Nullstellensatz established the one-to-one correspondence between radical ideals in $\mathbb{F}[z_1, \dots, z_n]$ and closed sets in $\mathbb{A}_{\mathbb{F}}^n$. We will study geometric properties like reducibility, dimension, singularity later in these lecture notes. Before this, we turn to the computational aspect of affine algebraic geometry, to see how to explicitly compute objects like $I_1 \cap I_2$ and $\mathcal{Z}(I)$.

2.4.2 Gröbner basis

One-variable case

We see that ideal is the central concept for the algebraic side of classical algebraic geometry. An ideal can be generated by different generating sets, some may be redundant or complicated. In linear algebra, given a linear subspace $V = \text{span}\{v_1 \dots v_k\}$ we may use Gaussian elimination to find the linearly-independent basis of V or Gram-Schmidt process to find an orthonormal basis. For ideals, a “good basis” can also dramatically simplify algebraic geometry problems.

Example 2.12. *As a toy model, consider some univariate cases.*

- For example, $I = \langle x^3 - x - 1 \rangle$ in $R = \mathbb{Q}[x]$. Clearly, I consists of all polynomials in x proportional to $x^3 - x - 1$, and every nonzero element in I has the degree higher or equal than 3. So we say $B(I) = \{x^3 - x - 1\}$ is a “good basis” for I . $B(I)$ is useful: for any polynomial $F(x)$ in $\mathbb{Q}[x]$, polynomial division determines,

$$F(x) = q(x)(x^3 - x - 1) + r(x), \quad q(x), r(x) \in \mathbb{Q}[x], \deg r(x) < 3 \quad (2.81)$$

Hence $F(x)$ is in I if and only if the remainder r is zero. It also implies that $R/I = \text{span}_{\mathbb{Q}}\{[1], [x], [x^2]\}$.

- Consider $J = \langle x^3 - x^2 + 3x - 3, x^2 - 3x + 2 \rangle$. Is the naive choice $B(J) = \{f_1, f_2\} = \{x^3 - x^2 + 3x - 3, x^2 - 3x + 2\}$ a good basis? For instance, $f = f_1 - xf_2 = 2x^2 + x - 3$ is in I but it is proportional to neither f_1 nor f_2 . Polynomial division over this basis is not useful, since f 's degree is lower than f_1 , the only division reads,

$$f = 2f_2 + (7x - 7). \quad (2.82)$$

The remainder does not tell us the membership of f in I . Hence $B(J)$ does not characterize I or R/I , and it is not “good”. Note that $\mathbb{Q}[x]$ is a principal ideal domain (PID), any ideal can be generated by one polynomial. Therefore, use Euclidean algorithm (Algorithm 1) to find the greatest common factor of f_1 and f_2 ,

$$(x - 1) = \frac{1}{7}f_1(x) - \frac{x + 2}{7}f_2(x), \quad (x - 1)|f_1(x), (x - 1)|f_2(x) \quad (2.83)$$

Hence $J = \langle x - 1 \rangle$. We can check that $\tilde{B}(J) = \{x - 1\}$ is a “good” basis in the sense that Euclidean division over $\tilde{B}(J)$ solves membership questions of J and determined $R/J = \text{span}_{\mathbb{Q}}\{[1]\}$.

Recall that in (2.53), given inverse propagators D_1, \dots, D_7 , we need to solve the membership problem of $I = \langle D_1 \dots D_7 \rangle$ and compute R/I . However, in general, a set like $\{D_1 \dots D_7\}$ is not a “good basis”, in the sense that the polynomial division over this basis does not solve the membership problem or give a correct integrand basis (as we see previously). Since it is a multivariate problem, the polynomial ring R is not a PID and we cannot use Euclidean algorithm to find a “good basis”.

Look at Example 2.12 again. For the univariate case, there is a natural monomial order \prec from the degree,

$$1 \prec x \prec x^2 \prec x^3 \prec x^4 \prec \dots, \quad (2.84)$$

Algorithm 1 Euclidean division for greatest common divisor

```

1: Input:  $f_1, f_2, \deg f_1 \geq \deg f_2$ 
2: while  $f_2 \nmid f_1$  do
3:     polynomial division  $f_1 = qf_2 + r$ 
4:      $f_1 := f_2$ 
5:      $f_2 := r$ 
6: end while
7: return  $f_2$  (gcd)

```

and all monomials are sorted. For any polynomial F , define the *leading term*, $\text{LT}(F)$ to be the highest monomial in F by this order (with the coefficient). For multivariate cases, the degree criterion is not fine enough to sort all monomials, so we need more general monomial orders.

Definition 2.13. Let M be the set of all monomials with coefficients 1, in the ring $R = \mathbb{F}[z_1, \dots, z_n]$. A monomial order \prec of R is an ordering on M such that,

1. \prec is a total ordering, which means any two different monomials are sorted by \prec .
2. \prec respects monomial products, i.e., if $u \prec v$ then for any $w \in M$, $uw \prec vw$.
3. $1 \prec u$, if $u \in M$ and u is not constant.

There are several important monomial orders. For the ring $\mathbb{F}[z_1, \dots, z_n]$, we use the convention $1 \prec z_n \prec z_{n-1} \prec \dots \prec z_1$ for all monomial orders. Given two monomials, $g_1 = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ and $g_2 = z_1^{\beta_1} \dots z_n^{\beta_n}$, consider the following orders:

- Lexicographic order (*lex*). First compare α_1 and β_1 . If $\alpha_1 < \beta_1$, then $g_1 \prec g_2$. If $\alpha_1 = \beta_1$, we compare α_2 and β_2 . Repeat this process until for certain α_i and β_i the tie is broken.
- Degree lexicographic order (*grlex*). First compare the total degrees. If $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, then $g_1 \prec g_2$. If total degrees are equal, we compare $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \dots$ until the tie is broken, like *lex*.
- Degree reversed lexicographic order (*grevlex*). First compare the total degrees. If $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, then $g_1 \prec g_2$. If total degrees are equal, we compare α_n and β_n . If $\alpha_n < \beta_n$, then $g_1 \succ g_2$ (reversed!). If $\alpha_n = \beta_n$, then we further compare $(\alpha_{n-1}, \beta_{n-1}), (\alpha_{n-2}, \beta_{n-2}) \dots$ until the tie is broken, and use the reversed result.
- Block order. This is the combination of *lex* and other orders. We separate the variables into k blocks, say,

$$\{z_1, z_2, \dots, z_n\} = \{z_1, \dots, z_{s_1}\} \cup \{z_{s_1+1}, \dots, z_{s_2}\} \dots \cup \{z_{s_{k-1}+1}, \dots, z_n\}. \quad (2.85)$$

Furthermore, define the monomial order for variables in each block. To compare g_1 and g_2 , first we compare the first block by the given monomial order. If it is a tie, we compare the second block... until the tie is broken.

Example 2.14. Consider $\mathbb{Q}[x, y, z]$, $z \prec y \prec x$. We sort all monomials up to degree 2 in *lex*, *grlex*, *grevlex* and the block order $[x] \succ [y, z]$ with *grevlex* in each block. This can be done by the following MATHEMATICA code:

$F = 1 + x + x^2 + y + xy + y^2 + z + xz + yz + z^2;$

$\text{MonomialList}[F, \{x, y, z\}, \text{Lexicographic}]$

$\text{MonomialList}[F, \{x, y, z\}, \text{DegreeLexicographic}]$

$\text{MonomialList}[F, \{x, y, z\}, \text{DegreeReverseLexicographic}]$

$\text{MonomialList}[F, \{x, y, z\}, \{\{1, 0, 0\}, \{0, 1, 1\}, \{0, 0, -1\}\}]$

and the output is,

$\{x^2, xy, xz, x, y^2, yz, y, z^2, z, 1\}$

$\{x^2, xy, xz, y^2, yz, z^2, x, y, z, 1\}$

$\{x^2, xy, y^2, xz, yz, z^2, x, y, z, 1\}$

$\{x^2, xy, xz, x, y^2, yz, z^2, y, z, 1\}$

Note that for *lex*, $x \succ y^2$, $y \succ z^2$ since we first compare the power of x and the y . The total degree is not respected in this order. On the other hand, *grlex* and *grevlex* both consider the total degree first. The difference between *grlex* and *grevlex* is that, $xz \succ_{\text{grlex}} y^2$ while $xz \prec_{\text{grevlex}} y^2$. So *grevlex* tends to set monomials with more variables, lower, in the list of monomials with a fixed degree. This property is useful for computational algebraic geometry. Finally, for this block order, $x \succ y^2$ since x 's degrees are compared first. But $y \prec z^2$, since $[y, z]$ block is in *grevlex*.

With a monomial order, we define the leading term as the highest monomial (with coefficient) of a polynomial in this order. Back to the second part of Example 2.12,

$$\text{LT}(f_1) = x^3 \quad \text{LT}(f_2) = x^2, \quad \text{LT}(x - 1) = x \quad (2.86)$$

The key observation is that although $x - 1 \in J$, its leading term is not divisible by the leading term of either f_1 or f_2 . This makes polynomial division unusable and $\{f_1, f_2\}$ is not a “good basis”. This leads to the concept of Gröbner basis.

Gröbner basis

Definition 2.15. For an ideal I in $\mathbb{F}[z_1, \dots, z_n]$ with a monomial order, a Gröbner basis $G(I) = \{g_1, \dots, g_m\}$ is a generating set for I such that for each $f \in I$, there always exists $g_i \in G(I)$ such that,

$$\text{LT}(g_i) \mid \text{LT}(f). \quad (2.87)$$

We can check that for the ideal J in Example 2.12, $\{f_1, f_2\}$ is not a Gröbner basis with respect to the natural order, while $\{x - 1\}$ is.

Multivariate polynomial division

To harness the power of Gröbner basis we need the multivariate division algorithm, which is a generalization of univariate Euclidean algorithm (Algorithm 2). The basic procedure is that: given a polynomial F and a list of k polynomials f_i 's, if $\text{LT}(F)$ is divisible by some $\text{LT}(f_i)$, then remove $\text{LT}(F)$ by subtracting a multiplier of f_i . Otherwise move $\text{LT}(F)$ to the remainder r . The output will be

$$F = q_1 f_1 + \dots q_k f_k + r, \quad (2.88)$$

where r consists of monomials cannot be divided by any $\text{LT}(f_i)$. Let $B = \{f_1, \dots, f_k\}$, we denote \overline{F}^B as the remainder r . Recall that the one-loop OPP integrand reduction and

Algorithm 2 Multivariate division algorithm

```

1: Input:  $F, f_1 \dots f_k, \succ$ 
2:  $q_1 := \dots := q_k = 0, r := 0$ 
3: while  $F \neq 0$  do
4:    $\text{reductionstatus} := 0$ 
5:   for  $i = 1$  to  $k$  do
6:     if  $\text{LT}(f_i) \mid \text{LT}(F)$  then
7:        $q_i := q_i + \frac{\text{LT}(F)}{\text{LT}(f_i)}$ 
8:        $F := F - \frac{\text{LT}(F)}{\text{LT}(f_i)} f_i$ 
9:        $\text{reductionstatus} := 1$ 
10:      break
11:    end if
12:  end for
13:  if  $\text{reductionstatus} = 0$  then
14:     $r := r + \text{LT}(F)$ 
15:     $F := F - \text{LT}(F)$ 
16:  end if
17: end while
18: return  $q_1 \dots q_k, r$ 

```

the naive trial of two-loop integrand reduction are very similar to this algorithm.

Note that for a general list of polynomials, the algorithm has two drawbacks: (1) the remainder r depends on the order of the list, $\{f_1, \dots, f_n\}$ (2) if $F \in \langle f_1 \dots f_n \rangle$, the algorithm may not give a zero remainder r . These made the previous two-loop integrand reduction unsuccessful. Gröbner basis eliminates these problems.

Proposition 2.16. *Let $G = \{g_1, \dots, g_m\}$ be a Gröbner basis in $\mathbb{F}[z_1, \dots, z_n]$ with the monomial order \succ . Let r be the remainder of the division of F by G , from Algorithm 2.*

1. r does not depend on the order of g_1, \dots, g_m .

2. If $F \in I = \langle g_1, \dots, g_m \rangle$, then $r = 0$.

Proof. If the division with different orders of g_1, \dots, g_n provides two remainder r_1 and r_2 . If $r_1 \neq r_2$, then $r_1 - r_2$ contains monomials which are not divisible by any $\text{LT}(g_i)$. But $r_1 - r_2 \in I$, this is a contradiction to the definition of Gröbner basis.

If $F \in I$, then $r \in I$. Again by the definition of Gröbner basis, if $r \neq 0$, $\text{LT}(r)$ is divisible by some $\text{LT}(g_i)$. This is a contradiction to multivariate division algorithm. \square

Then the question is: given an ideal $I = \langle f_1 \dots f_k \rangle$ in $\mathbb{F}[z_1, \dots, z_n]$ and a monomial order \succ , does the Gröbner basis exist and how do we find it? This is answered by Buchberger's Algorithm, which was presented in 1970s and marked the beginning of computational algebraic geometry.

Buchberger's Algorithm

Recall that for one-variable case, Euclidean algorithm (Algorithm 1) computes the gcd of two polynomials hence the Gröbner basis is given. The key step is to cancel leading terms of two polynomials. That inspires the concept of S-polynomial in multivariate cases.

Definition 2.17. Given a monomial order \succ in $R = \mathbb{F}[z_1, \dots, z_n]$, the S-polynomial of two polynomials f_i and f_j in R is,

$$S(f_i, f_j) = \frac{\text{LT}(f_j)}{\gcd(\text{LT}(f_i), \text{LT}(f_j))} f_i - \frac{\text{LT}(f_i)}{\gcd(\text{LT}(f_i), \text{LT}(f_j))} f_j. \quad (2.89)$$

Note that the leading terms of the two terms on the r.h.s cancel.

Theorem 2.18 (Buchberger). Given a monomial order \succ in $R = \mathbb{F}[z_1, \dots, z_n]$, Gröbner basis with respect to \succ exists and can be found by Buchberger's Algorithm (Algorithm 3).

Proof. See Cox, Little, O'Shea [CLO15]. \square

Algorithm 3 Buchberger algorithm

```

1: Input:  $B = \{f_1 \dots f_n\}$  and a monomial order  $\succ$ 
2:  $queue :=$  all subsets of  $B$  with exactly two elements
3: while  $queue \neq \emptyset$  do
4:    $\{f, g\} :=$  head of  $queue$ 
5:    $r := \overline{S(f, g)}^B$ 
6:   if  $r \neq 0$  then
7:      $B := B \cup r$ 
8:      $queue \leftarrow \{B_1, r\}, \dots, \{\text{last of } B, r\}$ 
9:   end if
10:  delete head of  $queue$ 
11: end while
12: return  $B$  (Gröbner basis)
```

The uniqueness of Gröbner basis is given via *reduced Gröbner basis*.

Definition 2.19. For $R = \mathbb{F}[z_1, \dots, z_n]$ with a monomial order \succ , a reduced Gröbner basis is a Gröbner basis $G = \{g_1, \dots, g_k\}$ with respect to \succ , such that

1. Every $\text{LT}(g_i)$ has the coefficient 1, $i = 1, \dots, k$.
2. Every monomial in g_i is not divisible by $\text{LT}(g_j)$, if $j \neq i$.

Proposition 2.20. For $R = \mathbb{F}[z_1, \dots, z_n]$ with a monomial order \succ , I is an ideal. The reduced Gröbner basis of I with respect to \succ , $G = \{g_1, \dots, g_m\}$, is unique up to the order of the list $\{g_1, \dots, g_m\}$. It is independent of the choice of the generating set of I .

Proof. See Cox, Little, O’Shea [CLO15, Chapter 2]. Note that given a Gröbner basis $B = \{h_1 \dots h_m\}$, the reduced Gröbner basis G can be obtained as follows,

1. For any $h_i \in B$, if $\text{LT}(h_j) \mid \text{LT}(h_i)$, $j \neq i$, then remove h_i . Repeat this process, and finally we get the *minimal basis* $G' \subset B$.
2. For every $f \in G'$, divide f towards $G' - \{f\}$. Then replace f by the remainder of the division. Finally, normalize the resulting set such that every polynomial has leading coefficient 1, and we get the reduced Gröbner basis G .

□

Note that Buchberger’s Algorithm reduces only one polynomial pair every time, more recent algorithms attempt to (1) reduce many polynomial pairs at once (2) identify the “unless” polynomial pairs *a priori*. Currently, the most efficient algorithms are Faugere’s F4 and F5 algorithms [Fau99, Fau02].

Usually we compute Gröbner basis by programs, for example,

- **MATHEMATICA** The embedded **GroebnerBasis** computes Gröbner basis by Buchberger’s Algorithm. The relation between Gröbner basis and the original generating set is not given. Usually, Gröbner basis computation in MATHEMATICA is not very fast.
- **MAPLE** Maple computes Gröbner basis by either Buchberger’s Algorithm or highly efficient F4 algorithm.
- **SINGULAR** is a powerful computer algebraic system [DGPS15] developed in University of Kaiserslautern. SINGULAR uses either Buchberger’s Algorithm or F4 algorithm to compute Gröbner basis.
- **MACAULAY2** is a sophisticated algebraic geometry program [GS], which orients to research mathematical problems in algebraic geometry. It contains Buchberger’s Algorithm and experimental codes of F4 algorithm.
- **Fgb package** [Fau10]. This is a highly efficient package of F4 and F5 algorithms by Jean-Charles Faugère. It has both MAPLE and C++ interfaces. Usually, it is faster than the F4 implement in MAPLE. Currently, coefficients of polynomials are restricted to \mathbb{Q} or \mathbb{Z}/p , in this package.

Example 2.21. Consider $f_1 = x^3 - 2xy$, $f_2 = x^2y - 2y^2 + x$. Compute the Gröbner basis of $I = \langle f_1, f_2 \rangle$ with grevlex and $x \succ y$ [CLO15].

We use Buchberger's Algorithm.

1. In the beginning, the list is $B := \{h_1, h_2\}$ and the pair set $P := \{(h_1, h_2)\}$, where $h_1 = f_1$, $h_2 = f_2$,

$$S(h_1, h_2) = -x^2, \quad h_3 := \overline{S(h_1, h_2)}^B = -x^2, \quad (2.90)$$

with the relation $h_3 = yh_1 - xh_2$.

2. Now $B := \{h_1, h_2, h_3\}$ and $P := \{(h_1, h_3), (h_2, h_3)\}$. Consider the pair (h_1, h_3) ,

$$S(h_1, h_3) = 2xy, \quad h_4 := \overline{S(h_1, h_3)}^B = 2xy, \quad (2.91)$$

with the relation $h_4 = -h_1 - xh_3$.

3. $B := \{h_1, h_2, h_3, h_4\}$ and $P := \{(h_2, h_3), (h_1, h_4), (h_2, h_4), (h_3, h_4)\}$. For the pair (h_2, h_3) ,

$$S(h_2, h_3) = -x + 2y^2, \quad h_5 := \overline{S(h_2, h_3)}^B = -x + 2y^2, \quad (2.92)$$

The new relation is $h_5 = -h_2 - yh_3$.

4. $B := \{h_1, h_2, h_3, h_4, h_5\}$ and

$$P := \{(h_1, h_4), (h_2, h_4), (h_3, h_4), (h_1, h_5), (h_2, h_5), (h_3, h_5), (h_4, h_5)\}. \quad (2.93)$$

For the pair (h_1, h_4) ,

$$S(h_1, h_4) = -4xy^2, \quad \overline{S(h_1, h_4)}^B = 0 \quad (2.94)$$

Hence this pair does not add information to Gröbner basis. Similarly, all the rests pairs are useless.

Hence the Groebner basis is

$$B = \{h_1, \dots, h_5\} = \{x^3 - 2xy, x^2y + x - 2y^2, -x^2, 2xy, 2y^2 - x\}. \quad (2.95)$$

Consider all the relations in intermediate steps, we determine the conversion between the old basis $\{f_1, f_2\}$ and B ,

$$\begin{aligned} h_1 &= f_1, & h_2 &= f_2, & h_3 &= f_1y - f_2x \\ h_4 &= -f_1(1 + xy) + f_2x^2, & h_5 &= -f_1y^2 + (xy - 1)f_2 \end{aligned} \quad (2.96)$$

Then we determine the reduced Gröbner basis. Note that $\text{LT}(h_3) \mid \text{LT}(h_1)$, $\text{LT}(h_4) \mid \text{LT}(h_2)$, so h_1 and h_2 are removed. The minimal Gröbner basis is $G' = \{h_3, h_4, h_5\}$. Furthermore,

$$\overline{h_3}^{\{h_4, h_5\}} = h_3, \quad \overline{h_4}^{\{h_3, h_5\}} = h_4, \quad \overline{h_5}^{\{h_3, h_4\}} = h_5 \quad (2.97)$$

so $\{h_3, h_4, h_5\}$ cannot be reduced further. The reduced Gröbner basis is

$$G = \{g_1, g_2, g_3\} = \{-h_3, \frac{1}{2}h_4, \frac{1}{2}h_5\} = \{x^2, xy, y^2 - \frac{1}{2}x\}. \quad (2.98)$$

The conversion relation is,

$$g_1 = -yf_1 + xf_2, \quad g_2 = -\frac{(1+xy)}{2}f_1 + \frac{1}{2}x^2f_2, \quad g_3 = -\frac{1}{2}y^2f_1 + \frac{1}{2}(xy-1)f_2. \quad (2.99)$$

MATHEMATICA finds G directly via **GroebnerBasis** $[\{x^3 - 2xy, x^2y - 2y^2 + x\}, \{x, y\}, \text{MonomialOrder} \rightarrow \text{DegreeReverseLexicographic}]$. However, it does not provide the conversion (2.99). This can be found by MAPLE or MACAULAY2.

As a first application of Gröbner basis, we can see some fractions can be easily simplified (like integrand reduction),

$$\begin{aligned} \frac{x^2}{(x^3 - 2xy)(x^2y - 2y^2 + x)} &= \frac{-yf_1 + xf_2}{f_1f_2} = -\frac{y}{f_2} + \frac{x}{f_1} \\ \frac{xy}{(x^3 - 2xy)(x^2y - 2y^2 + x)} &= \frac{-(1+xy)f_1/2 + x^2f_2/2}{f_1f_2} = -\frac{1+xy}{2f_2} + \frac{x^2}{2f_1} \\ \frac{y^2}{(x^3 - 2xy)(x^2y - 2y^2 + x)} &= \frac{h_5 + x/2}{f_1f_2} = \frac{x}{2f_1f_2} - \frac{y^2}{2f_2} + \frac{xy-1}{2f_1} \end{aligned} \quad (2.100)$$

In first two lines, we reduce a fraction with two denominators to fractions with only one denominator. In the last line, a fraction with two denominators is reduced to a fraction with two denominators but lower numerator degree ($y^2 \rightarrow x$). Higher-degree numerators can be reduced in the same way. Hence we conclude that all fractions $N(x, y)/(f_1f_2)$ can be reduced to,

$$\frac{1}{f_1f_2}, \quad \frac{x}{f_1f_2}, \quad \frac{y}{f_1f_2} \quad (2.101)$$

and fractions with fewer denominators. Note that even with this simple example, one-variable partial fraction method does not help the reduction.

We have some comments on Gröbner basis:

1. For $\mathbb{F}[z_1, \dots, z_n]$, the computation of polynomial division and Buchberger's Algorithm only used addition, multiplication and division in \mathbb{F} . No algebraic extension is needed. Let $\mathbb{F} \subset \mathbb{K}$ be a field extension. If $B = \{f_1, \dots, f_k\} \subset \mathbb{F}[z_1, \dots, z_n]$, then the Gröbner basis computation of B in $\mathbb{K}[x_1, \dots, x_n]$ produces a Gröbner basis which is still in $\mathbb{F}[z_1, \dots, z_n]$, irrelevant of the algebraic extension.
2. The form of a Gröbner basis and computation time dramatically depend on the monomial order. Usually, *grevlex* is the fastest choice while *lex* is the slowest. However, in some cases, Gröbner basis with *lex* is preferred. In these cases, we may instead consider some "midway" monomial order like block order, or convert a known *grevlex* basis to *lex* basis [FGLM93].
3. If all input polynomials are linear, then the reduced Gröbner basis is the *echelon form* in linear algebra.

2.4.3 Application of Gröbner basis

Gröbner basis is such a powerful tool that once it is computed, most computational problems on ideals are solved.

Ideal membership and fraction reduction

A Gröbner basis immediately solves the ideal membership problem. Given an $F \in R = \mathbb{F}[z_1, \dots, z_n]$, and $I = \langle f_1, \dots, f_k \rangle$. Let G be a Gröbner basis of I with a monomial order \succ . $F \in I$ if and only if $\overline{F}^G = 0$, i.e., the division of F towards G generates zero remainder (Proposition 2.16).

G also determined the structure of the quotient ring R/I (Definition 2.4). $f \sim g$ if and only if $f - g \in I$. The division of $f_1 - f_2$ towards G detects equivalent relations. In particular,

Proposition 2.22. *Let M be the set of all monic monomials in R which are not divisible by any leading term in G . Then the set,*

$$V = \{[p] | p \in M\} \quad (2.102)$$

is an \mathbb{F} -linear basis of R/I .

Proof. For any $F \in R$, \overline{F}^G consists of monomials which are not divisible by any leading term in G . Hence $[F]$ is a linear combination of finite elements in V .

Suppose that $\sum_j c_j [p_j] = 0$ and each p_j 's are monic monomials which are not divisible by leading terms of G . Then $\sum_j c_j p_j \in I$, but by the Algorithm 2. $\overline{\sum_j c_j p_j}^G = \sum_j c_j p_j$. So $\sum_j c_j p_j = 0$ in R and c_j 's are all zero. \square

As an application, consider fraction reduction for $N/(f_1 \dots f_k)$, where N is polynomial in R ,

$$\frac{N}{f_1 \dots f_k} = \frac{r}{f_1 \dots f_k} + \sum_{j=1}^k \frac{s_j}{f_1 \dots \hat{f}_j \dots f_k}. \quad (2.103)$$

The goal is to make r simplest, i.e., r should not contain any term which belongs to $I = \langle f_1, \dots, f_k \rangle$. We compute the Gröbner basis of I , $G = \{g_1, \dots, g_l\}$ and record the conversion relations $g_i = \sum_{j=1}^k f_j a_{ji}$ from the computation.

Polynomial division of N towards G gives,

$$N = r + \sum_{i=1}^l q_i g_i \quad (2.104)$$

where r is the remainder. The result,

$$\frac{N}{f_1 \dots f_k} = \frac{r}{f_1 \dots f_k} + \sum_{j=1}^k \frac{(\sum_{i=1}^l a_{ji} q_i)}{f_1 \dots \hat{f}_j \dots f_k}, \quad (2.105)$$

gives the complete reduction since by the properties of G , no term in r belongs to I . (2.105) solves integrand reduction problem for multi-loop diagrams. In practice, there are shortcuts to compute numerators like $(\sum_{i=1}^l a_{ji} q_i)$.

Solve polynomial equations with Gröbner basis

In general, it is very difficult to solve multivariate polynomial equations since variables are entangled. Gröbner basis characterizes the solution set and can also remove variable entanglements.

Theorem 2.23. *Let $f_1 \dots f_k$ be polynomials in $R = \mathbb{F}[x_1, \dots, x_n]$ and $I = \langle f_1 \dots f_k \rangle$. Let $\bar{\mathbb{F}}$ be the algebraic closure of \mathbb{F} . The solution set in $\bar{\mathbb{F}}$, $\mathcal{Z}_{\bar{\mathbb{F}}}(I)$ is finite, if and only if R/I is a finite dimensional \mathbb{F} -linear space. In this case, the number of solutions in $\bar{\mathbb{F}}$, counted with multiplicity, equals $\dim_{\mathbb{F}}(R/I)$.*

Proof. See Cox, Little, O’Shea [CLO98]. The rigorous definition of multiplicity is given in the next chapter, Definition 3.22. \square

Note that again, we distinguish \mathbb{F} and its algebraic closure $\bar{\mathbb{F}}$, since we do not need computations in $\bar{\mathbb{F}}$ to count total number of solutions in $\bar{\mathbb{F}}$. $\dim_{\mathbb{F}}(R/I)$ can be obtained by counting all monomials not divisible by $\text{LT}(G(I))$, leading terms of the Gröbner basis. Explicitly, $\dim_{\mathbb{F}}(R/I)$ is computed by **vdim** of SINGULAR.

Example 2.24. *Consider $f_1 = -x^2 + x + y^2 + 2$, $f_2 = x^3 - xy^2 - 1$. Determine the number of solutions $f_1 = f_2 = 0$ in \mathbb{C}^2 .*

Compute the Gröbner basis for $\{f_1, f_2\}$ in grevlex with $x \succ y$, we get,

$$G = \{y^2 + 3x + 1, x^2 + 2x - 1\}. \quad (2.106)$$

Then $\text{LT}(G) = \{y^2, x^2\}$. Then M in Proposition 2.22 is clearly $\{1, x, y, xy\}$. The linear basis for $\mathbb{Q}[x, y]/\langle f_1, f_2 \rangle$ is $\{[1], [x], [y], [xy]\}$. Therefore there are 4 solutions in \mathbb{C}^2 . Note that Bézout’s theorem would give the number $2 \times 3 = 6$. However, we are considering the solutions in affine space, so there are $6 - 4 = 2$ solutions at infinity. Another observation is that the second polynomial in G contains only x , so the variable entanglement disappears and we can first solve for x and then use x -solutions to solve y . This idea will be developed in the next topic, elimination theory.

Example 2.25 (Sudoku). *Sudoku is a popular puzzle with 9×9 spaces. The goal is to fill in digits from $1, 2, \dots, 9$, such that each row, each column and each 3×3 sub-box contain digits 1 to 9. See two Sudoku problems in Figure 2.6.*

Typically people solve Sudoku with backtracking algorithm: try to fill in as many digits as possible, and if there is no way to proceed then go one step back. It can be easily implemented in computer codes, and usually it is very efficient. Here we introduce solving Sudoku by Gröbner basis. This method is not the most efficient way, however, besides finding a solution, it illustrates the global structure of solutions.

We convert this puzzle to an algebraic problem. Name the digit on i -th row and j -th column as x_{ij} . x_{ij} must be in $\{1, \dots, 9\}$. Let,

$$F(x) = (x - 1)(x - 2) \dots (x - 9). \quad (2.107)$$

					8			4
	9				6		5	
6	5	2				1		
7					4		3	
			3	1	9			
	1		2					9
		1				9	6	7
	7		6				8	
4			9					

5	3			7				
6			1		5			
	9	8					6	
8								3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Figure 2.6: two Sudoku puzzles

So there are 81 equations, $F(x_{ij}) = 0$. Two spaces in the same row, or in the same column, or in the same sub-box, cannot contain the same digit. For example, $x_{11} \neq x_{12}$. Note this is not an equality, how do we write an algebraic equation to describe this constraint?

The standard trick to “differentiate” polynomials. Consider $F(y) - F(x)$, where x and y refer to two boxes that cannot contain the same digit. $F(y) - F(x)$ must be proportional to $y - x$.

$$\frac{F(y) - F(x)}{y - x} = g(x, y). \quad (2.108)$$

where $g(x, y)$ is a polynomial. It is clearly that when $y \neq x$, $g(x, y) = 0$. On the other hand, from the Taylor series,

$$F(y) - F(x) = (y - x) \left(F'(x) + \frac{1}{2}(y - x)F''(x) + \dots \right) = (y - x)g(x, y). \quad (2.109)$$

If $g(x, y) = 0$ but $y = x$, then $F'(x) = 0$. However $F(x)$ has no multiple root, that means $F(x)$ and $F'(x)$ cannot be both zero. So if $g(x, y) = 0$ then $y \neq x$. There are 810 such equations like $g(x_{11}, x_{12}) = 0$. Then with the known input information in Sudoku, we have a polynomial equation system.

For the first Sudoku, there are $81 + 810 + 27 = 918$ equations. It is really a large system with high degree polynomials. Amazingly, we can still solve it by Gröbner basis. Using **slimgb** command in SINGULAR, and the number field $\mathbb{Z}/11$, this sudoku is solved on a laptop computer with in about 4.9 seconds. The output Gröbner basis is linear and gives the unique solution of the Sudoku (Figure 2.7). For Sudoku 2, there are 919 equations. SINGULAR takes about 5.1 seconds on a laptop to get Gröbner basis ,

$$G = \{x_{58}^2 - 4, x_{11} - 5, x_{12} - 3, x_{13} - 4, x_{14} - 6, x_{15} - 7, x_{16} - 8, x_{17} - 9, x_{18} - 1, x_{19} - 2, \\ x_{21} - 6, x_{22} - 7, x_{23} - 2, x_{24} - 1, x_{25} - 9, x_{26} - 5, x_{27} - 3, x_{28} - 4, x_{29} - 8, x_{31} - 1, x_{32} - 9, \\ x_{33} - 8, x_{34} - 3, x_{35} - 4, x_{36} - 2, x_{37} - 5, x_{38} - 6, x_{39} - 7, x_{41} - 8, x_{42} + 9x_{58} - 9, \\ x_{43} + 9x_{58} - 2, x_{44} - 7, x_{45} + 3x_{58} - 11, x_{46} - 1, x_{47} - 4, x_{48} + x_{58} - 11, x_{49} - 3, x_{51} - 4,$$

1	3	7	5	9	8	6	2	4
8	9	4	1	2	6	7	5	3
6	5	2	7	4	3	1	9	8
7	6	9	8	5	4	2	3	1
2	4	8	3	1	9	5	7	6
5	1	3	2	6	7	8	4	9
3	2	1	4	8	5	9	6	7
9	7	5	6	3	1	4	8	2
4	8	6	9	7	2	3	1	5

Figure 2.7: Sudoku with unique solution, which is determined by Gröbner basis.

$$\begin{aligned}
& x_{52} + 2x_{58} - 9, x_{53} + 2x_{58} - 2, x_{54} - 8, x_{55} + 8x_{58} - 11, x_{56} - 3, x_{57} - 7, x_{59} - 1, x_{61} - 7, \\
& x_{62} - 1, x_{63} - 3, x_{64} - 9, x_{65} - 2, x_{66} - 4, x_{67} - 8, x_{68} - 5, x_{69} - 6, x_{71} - 9, x_{72} - 6, x_{73} - 1, \\
& x_{74} - 5, x_{75} - 3, x_{76} - 7, x_{77} - 2, x_{78} - 8, x_{79} - 4, x_{81} - 2, x_{82} - 8, x_{83} - 7, x_{84} - 4, x_{85} - 1, \\
& x_{86} - 9, x_{87} - 6, x_{88} - 3, x_{89} - 5, x_{91} - 3, x_{92} - 4, x_{93} - 5, x_{94} - 2, \\
& x_{95} - 8, x_{96} - 6, x_{97} - 1, x_{98} - 7, x_{99} - 9\}.
\end{aligned} \tag{2.110}$$

Note that the new feature is that G contains a quadratic polynomial, which means the solution for this sudoku is not unique. From leading term counting, there are 2 solutions. Explicitly, solve the first equation

$$x_{58}^2 = 4 \pmod{11}, \tag{2.111}$$

and we get two solutions, $x_{58} = 2$ or $x_{58} = 9$. Afterwards, we get two complete solutions (Figure 2.8).

Elimination theory

We already see that Gröbner basis can remove variable entanglement, here we study this property via elimination theory,

Theorem 2.26. *Let $R = \mathbb{F}[y_1, \dots, y_m, z_1, \dots, z_n]$ be a polynomial ring and I be an ideal in R . Then $J = I \cap \mathbb{F}[z_1, \dots, z_n]$, the elimination ideal, is an ideal of $\mathbb{F}[z_1, \dots, z_n]$. J is generated by $G(I) \cap \mathbb{F}[z_1, \dots, z_n]$, where $G(I)$ is the Gröbner basis of I in lex order with $y_1 \succ y_2 \dots \succ y_m \succ z_1 \succ z_2 \dots \succ z_n$.*

Proof. See Cox, Little and O'Shea [CLO15]. □

Note that elimination ideal J tells the relations between $z_1 \dots z_n$, without the interference with y_i 's. In this sense, y_i 's are “eliminated”. It is very useful for studying polynomial equation system. In practice, Gröbner basis in lex may involve heavy computations. So frequently, we use block order instead, $[y_1, \dots, y_m] \succ [z_1, \dots, z_n]$ while in each block $grevlex$ can be applied.

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	2	6	7	5	1	4	9	3
4	5	9	8	6	3	7	2	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Figure 2.8: Sudoku with multiple solutions, determined by Gröbner basis.

Eliminate theory applies in many scientific directions, for example, it transfers tree-level scattering equations (CHY formalism) [CHY14c, CHY14b, CHY14a, CHY15a, CHY15b] with n particles, in $(n - 3)$ variables, to a univariate polynomial equation [DG15]. Here we give a simple example in IMO,

Example 2.27 (International Mathematical Olympiad, 1961/1).

Problem Solve the system of equations:

$$\begin{aligned} x + y + z &= a \\ x^2 + y^2 + z^2 &= b^2 \\ xy &= z^2 \end{aligned} \tag{2.112}$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.

Solution The tricky part is the condition for positive distinct x, y, z . Now with Gröbner basis this problem can be solved automatically.

First, eliminate x, y by Gröbner basis in lex with $x \succ y \succ z$. For example, in MATHEMATICA

GroebnerBasis[- $a + x + y + z$, - $b^2 + x^2 + y^2 + z^2$, $xy - z^2$], { x, y, z },
MonomialOrder \rightarrow **Lexicographic**, **CoefficientDomain** \rightarrow **RationalFunctions**

and the resulting Gröbner basis is,

$$G = \{a^2 - 2az - b^2, -a^4 + y(2a^3 + 2ab^2) + 2a^2b^2 - 4a^2y^2 - b^4, a^2 - 2ax - 2ay + b^2\}. \tag{2.113}$$

The first element is in $\mathbb{Q}(a, b)[z]$, hence it generates the elimination ideal. Solve this equation, we get,

$$z = \frac{a^2 - b^2}{2a}. \tag{2.114}$$

Then eliminate y, z by Gröbner basis in lex with $z \succ y \succ x$. We get the equation,

$$a^4 + x(-2a^3 - 2ab^2) - 2a^2b^2 + 4a^2x^2 + b^4 = 0. \quad (2.115)$$

To make sure x is real we need the discriminant,

$$-4a^2(a^2 - 3b^2)(3a^2 - b^2) \geq 0. \quad (2.116)$$

Similarly, to eliminate x, z , we use lex with $z \succ x \succ y$ and get

$$a^4 + y(-2a^3 - 2ab^2) - 2a^2b^2 + 4a^2y^2 + b^4 = 0, \quad (2.117)$$

and the same real condition as (2.116). Note that x and y are both positive, if and only if x, y are real, $x + y > 0$ and xy . Hence positivity for x, y, z means,

$$\begin{aligned} z &= \frac{a^2 - b^2}{2a} > 0 \\ x + y &= a - z = a - \frac{a^2 - b^2}{2a} > 0 \end{aligned} \quad (2.118)$$

$$-4a^2(a^2 - 3b^2)(3a^2 - b^2) \geq 0. \quad (2.119)$$

which implies that,

$$a > 0, \quad b^2 < a^2 \leq 3b^2. \quad (2.120)$$

To ensure that x, y and z are distinct, we consider the ideal in $\mathbb{Q}[a, b, x, y, z]$.

$$J = \{-a + x + y + z, -b^2 + x^2 + y^2 + z^2, xy - z^2, (x - y)(y - z)(z - x)\}. \quad (2.121)$$

Note that to study the a, b dependence, we consider a and b as variables. Eliminate x, y, z , we have,

$$g(a, b) = (a - b)(a + b)(a^2 - 3b^2)^2(3a^2 - b^2) \in J. \quad (2.122)$$

If all the four generators in J are zero for some value of (a, b, x, y, z) , then $g(a, b) = 0$. Hence, if $g(a, b) \neq 0$, x, y and z are distinct in the solution. So it is clear that inside the region defined by (2.120), the subset set

$$a > 0, \quad b^2 < a^2 < 3b^2. \quad (2.123)$$

satisfies the requirement of the problem. On the other hand, if $a^2 = 3b^2$, explicitly we can check that x, y and z are not distinct in all solutions. Hence x, y, z in a solution are positive and distinct, if and only if $a > 0$ and $b^2 < a^2 < 3b^2$. With (2.114) and (2.115), it is trivial to obtain the solutions.

Intersection of ideals

In general, given two ideals I_1 and I_2 in $R = \mathbb{F}[z_1, \dots, z_n]$, it is very easy to get the generating sets for $I_1 + I_2$ and $I_1 I_2$. However, it is difficult to compute $I_1 \cap I_2$. Hence again we refer to Gröbner basis especially to elimination theory.

Proposition 2.28. *Let I_1 and I_2 be two ideals in $R = \mathbb{F}[z_1, \dots, z_n]$. Define J as the ideal*

generated by $\{tf | f \in I_1\} \cup \{(1-t)g | g \in I_2\}$ in $\mathbb{F}[t, z_1, \dots, z_n]$. Then $I_1 \cap I_2 = J \cap R$, and the latter can be computed by elimination theory.

Proof. If $f \in I_1$ and $f \in I_2$, then $f = tf + (1-t)f \in J$. So $I_1 \cap I_2 \subset J \cap R$. On the other hand, if $F \in J \cap R$, then

$$F(t, z_1, \dots, z_n) = a(t, z_1, \dots, z_n)tf(z_1, \dots, z_n) + b(t, z_1, \dots, z_n)(1-t)g(z_1, \dots, z_n), \quad (2.124)$$

where $f \in I_1, g \in I_2$. Since $F \in R$, F is t independent. Plug in $t = 1$ and $t = 0$, we get,

$$F = a(1, z_1, \dots, z_n)f(z_1, \dots, z_n), \quad F = b(0, z_1, \dots, z_n)g(z_1, \dots, z_n). \quad (2.125)$$

Hence $F \in I_1 \cap I_2, J \cap R \subset I_1 \cap I_2$. \square

In practice, terms like tf and $(1-t)g$ increase degrees by 1, hence this elimination method may not be efficient. More efficient method is given by *syzygy* computation [CLO98, Chapter 5].

2.4.4 Basic facts of algebraic geometry in affine space II

In this subsection, we look closer at properties of algebraic sets and ideals. Consider $I = \{x^2 - y^2, x^3 + y^3 - z^2\}$ in $\mathbb{C}[x, y, z]$. From naive counting, $\mathcal{Z}(I)$ is a curve since there are 2 equations in 3 variables. However, the plot of $\mathcal{Z}(I)$ (Figure 2.9) looks like a line and a cusp curve. So $\mathcal{Z}(I)$ is *reducible*, in the sense that it can be decomposed into smaller algebraic sets. So we need the concept of *primary decomposition*.

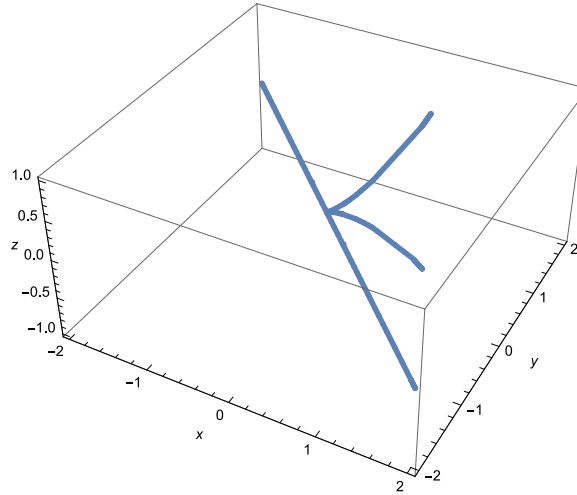


Figure 2.9: A reducible algebraic set (in blue), defined by $\mathcal{Z}(\{x^2 - y^2, x^3 + y^3 - z^2\})$.

Definition 2.29. An ideal I in a ring R is called *prime*, if $\forall ab \in I$ ($a, b \in R$) then $a \in I$ or $b \in I$. An ideal I in R is called *primary* if $ab \in I$ ($a, b \in R$) then $a \in I$ or $b^n \in I$, for some positive integer n .

A prime ideal must be a primary ideal. On the other hand,

Proposition 2.30. *If I is a primary ideal, then the radical of I , \sqrt{I} is a prime ideal.*

Proof. See Zariski and Samuel [ZS75a, Chapter 3]. \square

Note that $I = \{x^2 - y^2, x^3 + y^3 - z^2\}$ is not a prime ideal or primary ideal. Define $a = x - y$, $b = x + y$, clearly $ab \in I$, but $a \notin I$ and $b^n \notin I$ for any positive integer n . (The point $P = (2, 2, 4) \in \mathcal{Z}(I)$. If $(x + y)^n \in I$ then $(x + y)^n|_P = 0$. It is a contradiction.)

For another example, $J = \langle (x - 1)^2 \rangle$ in $\mathbb{C}[x]$ is primary but not prime. $\mathcal{Z}(J)$ contains only one point $\{1\}$ with the multiplicity 2. $(x - 1)(x - 1) \in J$ but $(x - 1) \notin J$. For these examples, we see primary condition implies that the corresponding algebraic set cannot be decomposed to smaller algebraic sets, while prime condition further requires that the multiplicity is 1.

Theorem 2.31 (Lasker-Noether). *For an ideal I in $\mathbb{F}[z_1, \dots, z_n]$, I has the primary decomposition,*

$$I = I_1 \cap \dots \cap I_m, \quad (2.126)$$

such that,

- Each I_i is a primary ideal in $\mathbb{F}[z_1, \dots, z_n]$,
- $I_i \not\supset \cap_{j \neq i} I_j$,
- $\sqrt{I_i} \neq \sqrt{I_j}$, if $i \neq j$.

Although primary decomposition may not be unique, the radicals $\sqrt{I_i}$'s are uniquely determined by I up to orders.

Proof. See Zariski, Samuel [ZS75a, Chapter 4]. \square

Note that unlike Gröbner basis, primary decomposition is very sensitive to the number field. For an ideal $I \subset \mathbb{F}[z_1, \dots, z_n]$, $\mathbb{F} \subset \mathbb{K}$, the primary decomposition results of I in $\mathbb{F}[z_1, \dots, z_n]$ and $\mathbb{K}[z_1, \dots, z_n]$ can be different. Primary decomposition can be computed by MACAULAY2 or SINGULAR. However, the computation is heavy in general.

Primary decomposition was also used for studying string theory vacua [MHH12].

Example 2.32. *Consider $I = \{x^2 - y^2, x^3 + y^3 - z^2\}$. Use MACAULAY2 or SINGULAR, we find that, $I = I_1 \cap I_2$, where,*

$$I_1 = \langle z^2, x + y \rangle, \quad I_2 = \langle 2y^3 - z^2, x - y \rangle \quad (2.127)$$

Then $\sqrt{I_1} = \langle z, x + y \rangle$ is a prime ideal, where I_2 itself is prime.

When $I \subset \mathbb{F}[z_1, \dots, z_n]$ has a primary decomposition $I = I_1 \cap \dots \cap I_m$, $m > 1$, then $\mathcal{Z}_{\mathbb{F}}(I) = \mathcal{Z}_{\mathbb{F}}(I_1) \cup \dots \cup \mathcal{Z}_{\mathbb{F}}(I_m)$. Then algebraic set decomposed to the union of sub algebraic sets. We switch the study of reducibility to the geometric side.

Definition 2.33. *Let V be a nonempty closed set in $\mathbf{A}_{\mathbb{F}}$ in Zariski topology, V is irreducible, if V cannot be a union of two closed proper subsets of V .*

Proposition 2.34. *Let \mathbb{K} be an algebraic closed field. There is a one-to-one correspondence:*

$$\begin{array}{ccc} \text{prime ideals in } \mathbb{K}[z_1, \dots, z_n] & & \text{irreducible algebraic sets in } \mathbf{A}_{\mathbb{K}} \\ I & \longrightarrow & \mathcal{Z}_{\mathbb{K}}(I) \\ \mathcal{I}(V) & \longleftarrow & V \end{array} \quad (2.128)$$

Proof. (Sketch) This follows from Hilbert Nullstellensatz (2.78). \square

We call an irreducible Zariski closed set “affine variety”. Similar to primary decomposition of ideals, algebraic set has the following decomposition,

Theorem 2.35. *Let V be an algebraic set. V uniquely decomposes as the union of affine varieties, $V = V_1 \cup \dots \cup V_m$, such that $V_i \not\supset V_j$ if $i \neq j$.*

Proof. Let $I = \mathcal{I}(V)$. The primary decomposition determines that $I = I_1 \cap \dots \cap I_m$. Since I is a radical ideal, all I_i ’s are prime. Then $V = \mathcal{Z}(I) = \cap_{i=1}^m \mathcal{Z}(I_i)$. Each $\mathcal{Z}(I_i)$ is an affine variety. If $\mathcal{Z}(I_i) \supset \mathcal{Z}(I_j)$, then $I_i \subset I_j$ which is a violation of radical uniqueness of Lasker-Noether theorem.

If there are two decompositions, $V = V_1 \cup \dots \cup V_m = W_1 \cup \dots \cup W_l$. $V_1 = V_1 \cap (W_1 \cup \dots \cup W_l) = (V_1 \cap W_1) \cup \dots \cup (V_1 \cap W_l)$. Since V_1 is irreducible, V_1 equals some $V_1 \cap W_j$, without loss of generality, say $j = 1$. Then $V_1 \subset W_1$. By the same analysis $W_1 \subset V_i$ for some i . Hence $V_1 \subset V_i$ and so $i = 1$. We proved $W_1 = V_1$. Repeat this process, we see that the two decompositions are the same. \square

Example 2.36. *As an application, we use primary decomposition to find cut solutions of 4D double box in Table 2.1. It is quite messy to derive all unitarity solutions by brute force computation. In this situation, primary decomposition is very helpful.*

Use van Neerven-Vermaseren variables, the ideal $I = \langle D_1, \dots, D_7 \rangle$ decomposes as $I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6$.

$$\begin{aligned} I_1 &= \{2y_4 - t, s + 2y_2, -t + 2x_3 - 2x_4, y_3, \frac{s}{2} + y_1 + y_2, x_2 - \frac{s}{2}, x_1\}, \\ I_2 &= \{t + 2y_4, s + 2y_2, -t + 2x_3 + 2x_4, y_3, \frac{s}{2} + y_1 + y_2, x_2 - \frac{s}{2}, x_1\}, \\ I_3 &= \{s + t + 2y_2 + 2y_4, 2x_4 - t, x_3, y_3, \frac{s}{2} + y_1 + y_2, x_2 - \frac{s}{2}, x_1\}, \\ I_4 &= \{s + t + 2y_2 - 2y_4, t + 2x_4, x_3, y_3, \frac{s}{2} + y_1 + y_2, x_2 - \frac{s}{2}, x_1\}, \\ I_5 &= \{s + t + 2y_2 + 2y_4, x_4(2s + 2t) + y_4(2s + 2t) + st + t^2 + 4x_4y_4, \\ &\quad -t + 2x_3 - 2x_4, y_3, \frac{s}{2} + y_1 + y_2, x_2 - \frac{s}{2}, x_1\}, \\ I_6 &= \{s + t + 2y_2 - 2y_4, x_4(-2s - 2t) + y_4(-2s - 2t) + st + t^2 + 4x_4y_4, \\ &\quad -t + 2x_3 + 2x_4, y_3, \frac{s}{2} + y_1 + y_2, x_2 - \frac{s}{2}, x_1\}. \end{aligned} \quad (2.129)$$

Each I_i is prime and corresponds to a solution in Table 2.1. SINGULAR computes this primary decomposition in about 3.6 seconds on a laptop. In practice, the computation can be sped up if we first eliminate all RSPs.

Hence the unitarity solution set $\mathcal{Z}(I)$ consists of six irreducible solution sets $\mathcal{Z}(I_i)$, $i = 1 \dots 6$. Each one can be parametrized by a free parameter.

Algebra		Geometry
Ideal I in $\mathbb{F}[z_1, \dots, z_n]$		algebraic set $\mathcal{Z}(I)$
$I_1 \cap I_2$		$\mathcal{Z}(I_1 \cap I_2) = \mathcal{Z}(I_1) \cup \mathcal{Z}(I_2)$
$I_1 + I_2$		$\mathcal{Z}(I_1 + I_2) = \mathcal{Z}(I_1) \cap \mathcal{Z}(I_2)$
$I_1 \subset I_2$	\Rightarrow	$\mathcal{Z}(I_1) \supset \mathcal{Z}(I_2)$
prime ideal I	\Rightarrow	$\mathcal{Z}(I)$ (irreducible) variety
maximal ideal I	\Rightarrow	$\mathcal{Z}(I)$ is a point
Krull dimension of $\dim \mathbb{F}[z_1, \dots, z_n]/I$	$=$	$\dim \mathcal{Z}(I)$

Table 2.2: algebraic geometry dictionary

For a variety V , we want to define its dimension. Intuitively, we may test if V contains a point, a curve, a surface...? So the dimension of V is defined as the length of variety sequence in V ,

Definition 2.37. *The dimension of a variety V , $\dim V$, is the largest number n in all sequences $\emptyset \neq W_0 \subset W_1 \subset \dots \subset W_n \subset V$, where W_i 's are distinct varieties.*

On the algebraic side, let $V = \mathcal{Z}(I)$, where I is an ideal in $R = \mathbb{F}[z_1, \dots, z_n]$. Consider the quotient ring R/I . Roughly speaking, the remaining “degree of freedom” of R/I should be the same as $\dim V$. Krull dimension counts “the degree of freedom”,

Definition 2.38 (Krull dimension). *The Krull dimension of a ring S , is the largest number n in all sequences $p_0 \subset p_1 \subset \dots \subset p_n$, where p_i 's are distinct prime ideals in S .*

If for a prime ideal I , R/I has Krull dimension zero then I is a *maximal ideal*. A maximal ideal I in R is an ideal such that for any proper ideal $J \supset I$, $J = I$. I is a maximal ideal, if and only if R/I is a field. (R itself is not a maximal ideal of R). When \mathbb{F} is algebraically closed, then any maximal ideal I in $R = \mathbb{F}[z_1, \dots, z_n]$ has the form [CLO15],

$$I = \langle z_1 - c_1, \dots, z_n - c_n \rangle, \quad c_i \in \mathbb{F}. \quad (2.130)$$

Note that the point (c_1, \dots, c_n) is zero-dimensional, and $R/I = \mathbb{F}$ has Krull dimension 0. More generally,

Proposition 2.39. *If \mathbb{F} is algebraically closed and I a prime proper ideal of $R = \mathbb{F}[z_1, \dots, z_n]$. Then the Krull dimension of R/I equals $\dim \mathcal{Z}(I)$.*

Proof. See Hartshorne [Har77, Chapter 1]. Note that Krull dimension of R/I is different from the linear dimension $\dim_{\mathbb{F}} R/I$. \square

In summary, we have the algebra-geometry dictionary (Table 2.2), where the last two rows hold if \mathbb{F} is algebraically closed.

We conclude this section by an example which applies Gröbner basis, primary decomposition and dimension theory.

Example 2.40. (*Galois theory*) *Galois theory studies the symmetry of a field extension, $\mathbb{F} \subset \mathbb{K}$ by the Galois group $\text{Aut}(\mathbb{K}/\mathbb{F})$. Historically, Galois group of a polynomial is defined to be the permutation group of roots, such that algebraic relations are preserved. Galois*

completely determined if a polynomial equation can be solved by radicals. In practice, given a polynomial to find its Galois group may be difficult. Here we introduce an automatic method of computing Galois group.

For example, consider the polynomial $f(x) = x^4 + 3x + 3$ in $\mathbb{Q}[x]$. It is irreducible over $\mathbb{Q}[x]$ and contains no multiple root in \mathbb{C} . We denote the four distant roots as x_1, x_2, x_3, x_4 . To ensure that these variables are distant, we use a classic trick in algebraic geometry: auxiliary variable. Introduce a new variable w , define that

$$I = \langle f(x_1), f(x_2), f(x_3), f(x_4), w(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) - 1 \rangle. \quad (2.131)$$

It is clear that in $\mathbb{C}[x_1, x_2, x_3, x_4, w]$, $\mathcal{Z}(I)$ is a finite set (for example via Gröbner basis computation.) The four variables must be distinct on the solution set, because of the last generator in (2.131). Back to $\mathbb{Q}[x_1, x_2, x_3, x_4, w]$, we want to find more algebraic relations over \mathbb{Q} which are “consistent” with I . That is to find a maximal ideal J in $\mathbb{Q}[x_1, x_2, x_3, x_4, w]$, $I \subset J$. In practice, we use primary decomposition and find that in $\mathbb{Q}[x_1, x_2, x_3, x_4, w]$,

$$I = I_1 \cap I_2 \cap I_3, \quad (2.132)$$

where explicitly each I_i is prime. Since $\dim_{\mathbb{Q}}(\mathbb{Q}[x_1, x_2, x_3, x_4, w]/I)$ is finite,

$$\dim_{\mathbb{Q}}(\mathbb{Q}[x_1, x_2, x_3, x_4, w]/I_1) < \infty. \quad (2.133)$$

I_1 is prime hence $\mathbb{Q}[x_1, x_2, x_3, x_4, w]/I_1$ has no zero divisor. A finite-dimensional \mathbb{Q} -algebra with no zero divisor must be a field. Hence I_1 is a maximal ideal of $\mathbb{Q}[x_1, x_2, x_3, x_4, w]$.

Compute the Groebner basis of I_1 with the block order $[w] \succ [x_1, x_2, x_3, x_4]$, we have

$$\begin{aligned} G(I_1) = \{ & x_1 + x_2 + x_3 + x_4, 2x_4^2 + 2x_2x_4 + 2x_3x_4 + x_2 + x_3 - 3, 2x_3x_2 + x_2 + x_3 + 3, \\ & x_2^2 - x_2 + x_3^2 - x_3, 4x_4^3 - 2x_4^2 + 6x_4 + 5x_2 + 5x_3 + 9, 2x_4x_3^2 + x_3^2 + 2x_4^2x_3 - 3x_3 + x_4^2 - 3x_4 - 3, \\ & 4x_3^3 - 2x_3^2 + x_3 - 5x_2 + 9, 315w - 2x_3^2 - 4x_4x_3 - 2x_4^2 + 3 \} \end{aligned} \quad (2.134)$$

Except the last one, polynomials in $G(I_1)$ provides all the algebraic relations over \mathbb{Q} of the four roots. Note that some relations are trivial like $x_1 + x_2 + x_3 + x_4 = 0$ which comes from coefficients of $f(x)$. Some relations like $2x_3x_2 + x_2 + x_3 + 3 = 0$, are nontrivial.

Consider all 24 permutations of (x_1, x_2, x_3, x_4) , we find the 8 of them preserves algebraic relations in $G(I_1)$, explicitly,

$$\begin{aligned} & (x_1, x_2, x_3, x_4), (x_1, x_3, x_2, x_4), (x_2, x_1, x_4, x_3), (x_3, x_1, x_4, x_2), \\ & (x_2, x_4, x_1, x_3), (x_3, x_4, x_1, x_2), (x_4, x_2, x_3, x_1), (x_4, x_3, x_2, x_1). \end{aligned} \quad (2.135)$$

Hence Galois group of the $x^4 + 3x + 3$ is the dihedral group D_4 . Clearly, this process applies to all irreducible polynomials without multiple root.

Note that $\mathbb{Q}[x_1, x_2, x_3, x_4, w]/I_1$ actually is the splitting field of this polynomial.

2.5 Multi-loop integrand reduction via Gröbner basis

With the knowledge of basic algebraic geometry, now multi-loop integrand reduction is almost a piece of cake. We apply Gröbner basis method [Zha12, MMOP12].

Consider the algorithm of direct integrand reduction (IR-D). Suppose that all terms with denominator set \mathcal{D} , $\{D_1, \dots, D_k\} \subsetneq \mathcal{D}$ are already reduced, then,

1. Collect all integrand terms with inverse propagators D_1, \dots, D_k , which include terms from Feynman rules and also terms from the integrand reduction of parent diagrams. Denote the sum as $N/(D_1, \dots, D_k)$.
2. Define $I = \langle D_1, \dots, D_k \rangle$. Compute the Gröbner basis of I in *grevlex*, $G(I) = \{g_1, g_2, \dots, g_m\}$.
3. Polynomial division $N = a_1 g_1 + \dots + a_m g_m + \Delta$. Use Gröbner basis convention relation, rewrite the division as $N = q_1 D_1 + \dots + q_k D_k + \Delta$.
4. Add $\Delta/(D_1 \dots D_k)$ to the final result. Keep terms

$$\frac{q_1}{\hat{D}_1 D_2 \dots D_k} + \frac{q_2}{D_1 \hat{D}_2 \dots D_k} + \dots + \frac{q_k}{D_1 D_2 \dots \hat{D}_k}, \quad (2.136)$$

for child diagrams.

Repeat this process, until all terms left are integrated to zero (like massless tadpoles, integral without loop momenta dependences).

Integrand reduction (IR-U) is more subtle. Again, Suppose that all diagrams with denominator set \mathcal{D} , $\{D_1, \dots, D_k\} \subsetneq \mathcal{D}$ are reduced, then,

1. Define $I = \langle D_1, \dots, D_k \rangle$. Compute the Gröbner basis of I in *grevlex* with numeric kinematics, $G(I) = \{g_1, g_2, \dots, g_m\}$.
2. Identify all degree-one polynomials in $G(I)$, and solve them linearly. The dependent variables are RSPs. Define J as the ideal obtained by eliminate all RSPs in I .
3. Make a numerator ansatz N in ISPs, with the power counting restriction from renormalization conditions. Divide N toward $G(J)$, the remainder Δ is the integrand basis.
4. Cut all propagators by $D_1 = \dots = D_k = 0$. Classify all solutions by the primary decomposition of J and get n irreducible solutions.
5. On the cut, compute the tree products summed over internal spins/helicities. Subtract all known parent diagrams on this cut. The result should be a list of n functions S_i , defined on each cut solution.
6. Fit coefficients of Δ from S_i 's.

We have some comments here:

- To make an integrand basis with undetermined coefficients, we only need Gröbner basis with numeric kinematic conditions.
- RSPs can be automatically found, because any degree-one polynomial in I should be a linear combination of degree-one polynomials in $G(I)$, via Algorithm 2. Hence linear algebra computation determines RSPs.
- Integrand basis should not contain RSPs. Furthermore, it is helpful to eliminate RSPs before the primary decomposition.
- If the cut solution is complicated, primary decomposition helps finding all of solutions. And in general, solution sets cannot be parameterized rationally before primary decomposition.

The key idea of these algorithms is that polynomial division via Gröbner basis provides the simplest integrand, in the sense that the resulting numerator does not contain any term which are divisible by denominators.

Back to our double box examples, we use algebraic geometry methods to automate most of the computations. Given 7 propagators in Van Neerven-Vermaseren variables, we use number field $\mathbb{F} = \mathbb{Q}(s, t)$, define the ideal $I = \langle D_1, D_2, \dots, D_7 \rangle$.

First, we determine the RSPs. Compute $G(I)$ in *grevlex*, with numeric kinematics, $t \rightarrow -3, s \rightarrow 1$. We find that $G(I)$ contains 4 linear polynomials,

$$\{y_3, \frac{1}{2} + y_1 + y_2, x_2 - \frac{1}{2}, x_1\} \subset G(I). \quad (2.137)$$

This allow us to define RSPs: we have 4 linear polynomials and 5 variables, pick up y_1 to be the free variable. And then we determined x_1, x_2, y_2, y_3 are RSPs. (If needed, the full RSP relations can be obtained from Groebner basis conversion.)

$$\begin{aligned} x_1 &= \frac{D_1 - D_2}{2}, & x_2 &= \frac{D_2 - D_3}{2} + \frac{s}{2}, \\ y_2 &= \frac{D_4 - D_6}{2} - \frac{s}{2} - y_1, & y_3 &= \frac{-D_6 + D_7}{2}. \end{aligned} \quad (2.138)$$

Then, we consider to eliminate RSPs. Define J to be an ideal in $\mathbb{F}[x_3, y_1, x_4, y_4]$, which is the ideal after RSP elimination. With numeric kinematics, the Gröbner basis of J in *grevlex* and $y_4 \succ x_4 \succ y_1 \succ x_3$ is,

$$\begin{aligned} G(J) = \{ & -4x_3^2 - 12x_3 + 4x_4^2 - 9, 20x_3y_1 + 4x_4y_4 + 6x_3 + 6y_1 + 9, -4y_1^2 - 12y_1 + 4y_4^2 - 9, \\ & 4x_3^2y_4 + 20x_4x_3y_1 + 12x_3y_4 + 6x_4y_1 + 6x_4x_3 + 9x_4 + 9y_4, \\ & 4x_4y_1^2 + 12x_4y_1 + 20x_3y_4y_1 + 6x_3y_4 + 9x_4 + 6y_4y_1 + 9y_4, 4x_3^2y_1^2 + 2x_3^2y_1 + 2x_3y_1^2 + 3x_3y_1, \\ & 80x_3^2y_1y_4 + 16x_3^2y_4 + 40x_3y_1y_4 + 18x_3y_4 - 6x_4y_1 + 24x_4x_3 - 9x_4 - 9y_4\}. \end{aligned} \quad (2.139)$$

Note that the first 3 polynomials are just equations in (2.56). However, the rest algebra relations in (2.139) are not obtained by the naive generalization of OPP method. So previously we got a redundant basis.

Consider the numerator in ISPs only,

$$N_{\text{dbox}} = \sum_m \sum_n \sum_\alpha \sum_\beta c'_{mn\alpha\beta} x_3^m y_1^n x_4^\alpha y_4^\beta, \quad (2.140)$$

where $c'_{mn\alpha\beta}$ are indeterminate coefficients. By renormalization condition, there 160 such c 's. Divide N_{dbox} by $G(I)$, we get the remainder,

$$\Delta_{\text{dbox}} = \sum_{(m,n,\alpha,\beta) \in S} c_{mn\alpha\beta} x_3^m y_1^n x_4^\alpha y_4^\beta, \quad (2.141)$$

where the index set S contains 32 elements,

$$\begin{aligned} & (0, 0, 0, 0), (1, 0, 0, 0), (2, 0, 0, 0), (3, 0, 0, 0), (4, 0, 0, 0), (0, 1, 0, 0), (1, 1, 0, 0), (2, 1, 0, 0), \\ & (3, 1, 0, 0), (4, 1, 0, 0), (0, 2, 0, 0), (1, 2, 0, 0), (0, 3, 0, 0), (1, 3, 0, 0), (0, 4, 0, 0), (1, 4, 0, 0), \\ & (0, 0, 1, 0), (1, 0, 1, 0), (2, 0, 1, 0), (3, 0, 1, 0), (0, 1, 1, 0), (0, 0, 0, 1), (1, 0, 0, 1), (2, 0, 0, 1), \\ & (3, 0, 0, 1), (4, 0, 0, 1), (0, 1, 0, 1), (1, 1, 0, 1), (0, 2, 0, 1), (1, 2, 0, 1), (0, 3, 0, 1), (1, 3, 0, 1). \end{aligned} \quad (2.142)$$

Note that the number of terms in Δ_{dbox} matches the number of independent relations from unitarity cuts. (2.142) is the integrand basis of the $4D$ double box. Of these 32 terms, the last 16 terms integrated to zero by Lorentz symmetry, so they are spurious terms.

In Example 2.36, we already used primary decomposition to find all unitarity-cut solutions. Note that there is shortcut" it is enough to consider the primary decomposition of J . On a laptop computer, it takes only 0.22 seconds to finish. Using (2.142) and $4D$ tree amplitudes, we can easily determine the double box integrand for (super)-Yang-Mills theory [BFZ12b, Zha12].

For $D = 4 - 2\epsilon$, we need to introduce μ variables,

$$\begin{aligned} l_i &= l_i^{[4]} + l_i^\perp, \quad i = 1, \dots, L, \\ \mu_{ij} &= -l_i^\perp \cdot l_j^\perp, \quad 1 \leq i \leq j \leq L. \end{aligned} \quad (2.143)$$

In this case, we have further simplification: $I = \langle D_1, \dots, D_k \rangle$ must be a prime ideal, hence it is not necessary to consider the primary decomposition of I [Zha12, BFZ13].

Example 2.41. Consider two-loop five-gluon pure Yang-Mills planar amplitude, with helicity $(+++++)$. Note that tree-level all-plus-helicity 5-gluon amplitude in Yang-Mills theory is zero, while the one-loop-level is finite. The two-loop amplitude is much more challenging. We used algebraic geometry method to compute this amplitude.

For the integrand, we use both IR-D and IR-U methods [BFZ13]. Note that this amplitude is well-define only with $D = 4 - 2\epsilon$. Repeat the integrand reduction process, we get all the diagrams with non-vanishing integrands in Figure 2.10 (and their permutations). For example, the box-pentagon diagram for this amplitude has a simple integrand,

$$\Delta_{431}(1^+, 2^+, 3^+, 4^+, 5^+) =$$

$$- \frac{i s_{12} s_{23} s_{45} F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \text{tr}_5} (\text{tr}_+(1345)(l_1 + k_5)^2 + s_{15} s_{34} s_{45}) \quad (2.144)$$

where

$$F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) = (D_s - 2)(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33}) + 16(\mu_{12}^2 - \mu_{11}\mu_{22}), \quad (2.145)$$

and $\mu_{33} = \mu_{11} + \mu_{22} + 2\mu_{12}$ and D_s is the dimension for internal states. [BFZ13].

$$\begin{aligned} \text{tr}_5 &= \text{tr}(\gamma_5 \not{k}_1 \not{k}_2 \not{k}_3 \not{k}_4) = [12]\langle 23 \rangle [34]\langle 41 \rangle - \langle 12 \rangle [23]\langle 34 \rangle [41]. \\ \text{tr}_\pm(abcd) &= \frac{1}{2} \text{tr}((1 \pm \gamma_5) \not{k}_a \not{k}_b \not{k}_c \not{k}_d), \end{aligned} \quad (2.146)$$

Results from IR-D and IR-U match each other. After getting these simple integrand, the complete integrals and final analytic result for this amplitude was obtained by differential equation method [GHL16].

All-plus two-loop five-gluon non-planar integrand and all-plus two-loop six-gluon integrand were also obtained by integrand reduction method [BMO15, BMP16].

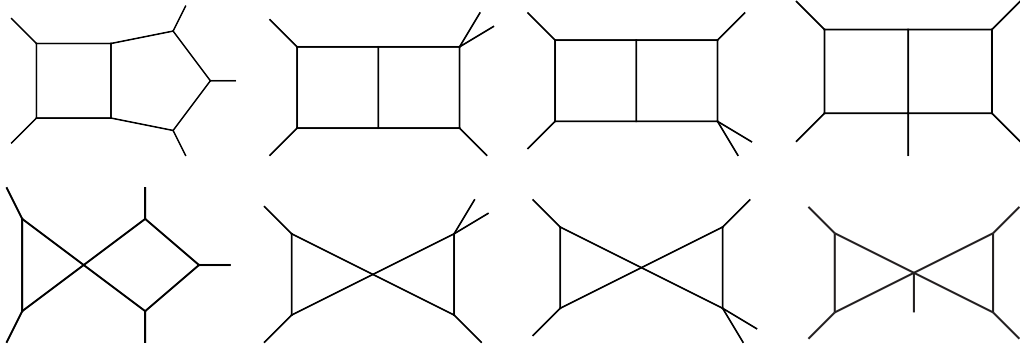


Figure 2.10: Nonzero diagrams from the integrand reduction, for (+ + + + +)-helicity two-loop five-gluon planar amplitude.

See [FH13, MMOP13, MPP16] for more example of CAG based integrand reductions.

2.6 Exercises

Exercise 2.1. Derive the integrand basis of a box diagram with $k_1^2 = k_2^2 = k_3^2 = k_4^2 = m^2$ and inverse propagators $D_1 = l_1^2$, $D_2 = (l_1 - k_1)^2$, $D_3 = (l_1 - k_1 - k_2)^2$ and $D_4 = (l_1 + k_4)^2$, via OPP approach [OPP07, OPP08].

Exercise 2.2 (Basic operations of ideals). I and J are two ideals in $\mathbb{F}[z_1, \dots, z_n]$. Define $I + J = \{f + g | f \in I, g \in J\}$ and IJ as the ideal generated by the set $\{fg | f \in I, g \in J\}$.

1. Prove that \sqrt{I} , $I + J$, $I \cap J$ are ideals.
2. Prove that $IJ \subset I \cap J$ and $\sqrt{IJ} = \sqrt{I \cap J}$.

3. Let $I = \langle y(y - x^2) \rangle$, $J = \langle xy \rangle$ in $\mathbb{Q}[x, y]$, determine $Z_{\mathbb{C}}(I + J)$, $Z_{\mathbb{C}}(I \cap J)$ and $Z_{\mathbb{C}}(IJ)$. Compute generating sets of $I + J$, $I \cap J$ and IJ . Is IJ the same as $I \cap J$ in this case? Compute generating sets of $\sqrt{I \cap J}$ and \sqrt{IJ} .

Exercise 2.3 (Hilbert's weak Nullstellensatz). Let $f_1(x) = 2x - 4x^2 + x^3$ and $f_2(x) = x^2 - 1$. Prove that as an ideal in $\mathbb{Q}[x]$, $\langle f_1, f_2 \rangle = \langle 1 \rangle$. Explicitly find two polynomials $h_1(x)$ and $h_2(x)$ in $\mathbb{Q}[x]$ such that,

$$h_1(x)f_1(x) + h_2(x)f_2(x) = 1. \quad (2.147)$$

(Hint: use Euclid's algorithm, Algorithm 1.)

Exercise 2.4 (Zariski topology). Prove (2.79) and (2.80).

$$\begin{aligned} \bigcap_i \mathcal{Z}(I_i) &= \mathcal{Z}\left(\bigcup_i I_i\right). \\ \mathcal{Z}(I_1) \bigcup \mathcal{Z}(I_2) &= \mathcal{Z}(I_1 I_2) = \mathcal{Z}(I_1 \cap I_2). \end{aligned} \quad (2.148)$$

Exercise 2.5 (Elimination theory).

1. Use computer software like MATHEMATICA, MAPLE, SINGULAR or MACAULAY2, to eliminate y and z from

$$I = \langle -x^3 - xz + y^2 - 1, x^2 + xz + y^2, xy + xz + y \rangle, \quad (2.149)$$

to get a equation in x only. How many common zeros are there for the three polynomials over \mathbb{C} ?

2. Use computer software to find the projection of the curve \mathcal{C} ,

$$\mathcal{C}: \quad x^2 + xy + z^2 = x^2 - zy - z^3 + 1 = 0, \quad (2.150)$$

on x - y plane.

Exercise 2.6 (Polynomial division via Gröbner basis). Let $f_1 = y^2 - x^3 - 1$, $f_2 = xy + y^2 + 1$ and $f_3 = y^2 + x - y$. Use MAPLE or MACAULAY to find the Gröbner basis $G = \{g_1, \dots, g_m\}$ and the conversion,

$$g_j = \sum_{i=1}^3 f_i a_{ij}. \quad (2.151)$$

Reduce the fraction $1/(f_1 f_2 f_3)$ as,

$$\frac{1}{f_1 f_2 f_3} = \frac{q_1}{f_2 f_3} + \frac{q_2}{f_1 f_3} + \frac{q_3}{f_1 f_2} \quad (2.152)$$

where q_1 , q_2 and q_3 are polynomials in x and y .

Exercise 2.7 (Primary decomposition). Use MACAULAY or SINGULAR to find the primary decomposition of $I = \langle xz - y^2, x^3 - yz \rangle$. Then parameterize each irreducible closed set.

Exercise 2.8 (Galois group and primary decomposition). Use the method in Example 2.40, to determine the Galois group of $x^4 - 10x^2 + 1$.

Exercise 2.9 (Integrand basis via Gröbner basis). Massless crossed box diagram is the two-loop diagram with $k_1^2 = k_2^2 = k_3^2 = k_4^2 = 0$ and inverse propagators $D_1 = l_1^2$, $D_2 = (l_1 - k_1)^2$, $D_3 = (l_1 - k_1 - k_2)^2$, $D_4 = l_2^2$, $D_5 = (l_2 - k_4)^2$, $D_6 = (l_1 + l_2 - k_1 - k_2 - k_4)^2$, $D_7 = (l_1 + l_2)^2$. Find the 4D integrand basis via Gröbner basis.

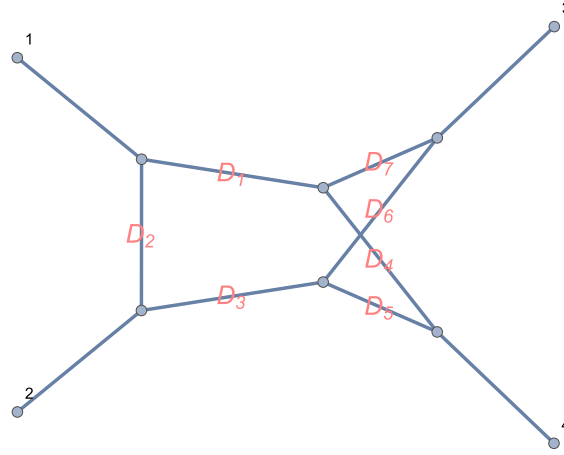


Figure 2.11: crossed box diagram

Exercise 2.10 (Fit integrand basis from unitarity cuts). Use the 4D double box integrand basis (2.142) to determine the double box integrand form of the 4D $(--++)$ and $(-+-+)$ helicity color-ordered amplitude in pure-Yang-Mills theory. (Hint: see [BFZ12b].)

Chapter 3

Unitarity Cuts and Several Complex Variables

3.1 Maximal unitarity

Besides the integrand reduction method for loop amplitudes, we can also consider the (generalized) unitarity with residue approach [BDDK94,BDDK95,BM96,BCF05,BCFW05]

$$A_n^{L\text{-loop}} = \sum_i c_i I_i + \text{rational terms} . \quad (3.1)$$

The set $\{I_k\}$ is the master integral (MI) basis, i.e., minimal linear basis of Feynman integrals. For example, for one-loop order, we have *scalar* box, triangle, bubble (and tadpole) integrals. The MI basis is usually a proper subset of the integrand basis like (2.15), since spurious terms are removed and integration-by-parts (IBP) identities are used.

Maximal unitarity method gets coefficients c_i 's for a scattering process, from contour integrals. (Usually contour integrals are simpler than Euclidean Feynman integrals.) Let k be the largest number of propagators for all integrals in MI basis. Suppose that there are $d(k)$ diagrams with exactly k propagators in the master integral list, $\mathcal{D}_1, \dots, \mathcal{D}_{d(k)}$.

Maximal unitarity method first separate (3.1) as,

$$A_n^{L\text{-loop}} = \sum_{\alpha=1}^{d(k)} \sum_j c_{\alpha,j} I_{\alpha,j} + (\text{simpler integrals}) + \text{rational terms} , \quad (3.2)$$

where for fixed α , $I_{\alpha,j}$'s stand for all master integrals associated with the diagram \mathcal{D}_α . “Simpler integrals” stands for integrals with fewer-than- k propagators.

The coefficients $c_{\alpha,j}$'s can be obtained by maximal unitarity as follows: Let the propagators of \mathcal{D}_α be D_1, \dots, D_k . For simplicity, we drop the index α . The cut equation is

$$D_1 = \dots = D_k = 0 , \quad (3.3)$$

which has m independent solutions. In algebraic geometry language, the ideal $I =$

$\langle D_1 \dots D_k \rangle$ has the primary decomposition,

$$I = I_1 \cap \dots \cap I_m. \quad (3.4)$$

Each independent solution is an (irreducible) variety, $V_i = \mathcal{Z}(I_i)$. For an *integer value* of the spacetime dimension D , we replace a generic Feynman integral by a contour integral,

$$\begin{aligned} \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N(l_1, \dots, l_L)}{D_1 \dots D_k} &\rightarrow \oint \frac{d^D l_1}{(2\pi i)^D} \cdots \frac{d^D l_L}{(2\pi i)^D} \frac{N(l_1, \dots, l_L)}{D_1 \dots D_k} \\ &= \sum_{i=1}^m \sum_b w_b^{(i)} \oint_{\mathcal{C}_b^{(i)}} \Omega^{(i)}(N). \end{aligned} \quad (3.5)$$

In the first line, we have a DL -fold contour integral. Part of the contour integrals serve as “holomorphic” Dirac delta functions in D_1, \dots, D_k , and the original integral becomes $(\dim V_i)$ -fold contour integrals on each V_i . The $\mathcal{C}_b^{[i]}$ ’s are non-trivial contours on V_i for this integrand, which consists of poles in the integrand and fundamental cycles of V_i . On each cut solution, the original numerator $N(l_1, \dots, l_L)$ becomes,

$$N(l_1, \dots, l_L)|_{V_i} = S^{(i)}. \quad (3.6)$$

where $S^{(i)}$ is the sum of products of tree amplitudes obtained from the maximal cut. In general, there may be several nontrivial contours on V_i , so for each one we set up a weight $w_b^{(i)}$ to be determined later.

We demand that if the original integral is zero, or can be reduce to integrals with fewer propagators by IBPs, the corresponding contour integral is zero. If

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{F(l_1, \dots, l_L)}{D_1 \dots D_k} = 0, \quad (F \text{ is spurious}), \quad (3.7)$$

or

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{F(l_1, \dots, l_L)}{D_1 \dots D_k} = (\text{simpler integrals}), \quad (\text{IBP relation}), \quad (3.8)$$

Then

$$\sum_{i=1}^m \sum_b w_b^{(i)} \oint_{\mathcal{C}_b^{(i)}} \Omega^{(i)}(F) = 0. \quad (3.9)$$

Spurious terms and IBPs fix $w_b^{(i)}$ ’s up to the normalization of master integrals. To extract the coefficients c_i ’s in (3.1), we can find a special set of weights $w_{b,j}^{(i)}$ such that,

$$c_j = \sum_{i=1}^m \sum_b w_{b,j}^{(i)} \oint_{\mathcal{C}_b^{(i)}} \Omega^{(i)}(S^{(i)}). \quad (3.10)$$

After getting all k -propagator master integrals’ coefficients, we repeat this process for $(k-1)$ propagator integrals. We need spurious terms and IBPs to fix the contour weights.

For example, consider the $4D$ massless four-point amplitude. (3.2) reads.

$$A_4^{1\text{-loop}} = c_{\text{box}} I_{\text{box}} + \dots \quad (3.11)$$

From (2.18), $D_1 = D_2 = D_3 = D_4 = 0$ has two solutions. Change the original integral to contour integrals,

$$\int \frac{d^4 l_1}{i\pi^2} \frac{1}{D_1 D_2 D_3 D_4} \rightarrow \frac{2}{t(s+t)} \oint \frac{dx_1 dx_2 dx_3 dx_4}{(2\pi i)^4} \frac{1}{D_1 D_2 D_3 D_4} = \begin{cases} \frac{1}{4st} & \text{on } V_1 \\ -\frac{1}{4st} & \text{on } V_2 \end{cases} \quad (3.12)$$

and

$$\int \frac{d^4 l_1}{i\pi^2} \frac{l \cdot \omega}{D_1 D_2 D_3 D_4} \rightarrow \frac{2}{t(s+t)} \oint \frac{dx_1 dx_2 dx_3 dx_4}{(2\pi i)^4} \frac{x_4}{D_1 D_2 D_3 D_4} = \begin{cases} \frac{1}{8s} & \text{on } V_1 \\ \frac{1}{8s} & \text{on } V_2 \end{cases} \quad (3.13)$$

We have two weights $\omega^{(1)}$ and $\omega^{(2)}$. From the contour integral of spurious term (3.13),

$$\omega^{(1)} \frac{1}{8s} + \omega^{(2)} \frac{1}{8s} = 0. \quad (3.14)$$

Hence $\omega^{(2)} = -\omega^{(1)}$. Normalize the weights for the scalar box integral,

$$\omega^{(1)} = 2st, \quad \omega^{(2)} = -2st. \quad (3.15)$$

Hence

$$\begin{aligned} c_{\text{box}} &= 2st \cdot \frac{2}{t(s+t)} \left(\oint_{V_1} \frac{dx_1 dx_2 dx_3 dx_4}{(2\pi i)^4} \frac{N}{D_1 D_2 D_3 D_4} - \oint_{V_2} \frac{dx_1 dx_2 dx_3 dx_4}{(2\pi i)^4} \frac{N}{D_1 D_2 D_3 D_4} \right) \\ &= \frac{1}{2} S^{(1)} + \frac{1}{2} S^{(2)}. \end{aligned} \quad (3.16)$$

which is the same as (2.21).

The two-loop maximal unitarity method was first invented in [KL12] for the $4D$ massless double box diagram, in an elegant way of determining all contours and corresponding contour weights. Afterwards, this method was generalized for the double box diagram with external massive legs [JKL12, JKL13b, JKL13a, CHL12].

In general, for multi-loop cases, the contour integrals are multivariate, and can be complicated in some cases. There are complicated issues with (3.5):

1. The solution set V_i is not a rational variety. For example, V_i can be an elliptic curve or a hyper-elliptic curve. Then contour integrals are not only residue computations, but also integrals over the fundamental cycles. Some of these cases are treated by maximal unitarity with complete elliptic integrals or hyper-elliptic integrals [SZ15, GZ15]. There is a rich algebraic geometry structure in this direction and these integrals are important for the LHC physics. But we are not going to cover this direction in these notes, since the background knowledge of algebraic curves needs to be introduced.
2. The residue is multivariate and Cauchy's formula does not work since the Jacobian at the pole is zero. For example, the $4D$ slashed box diagram and the $4D$ triple box

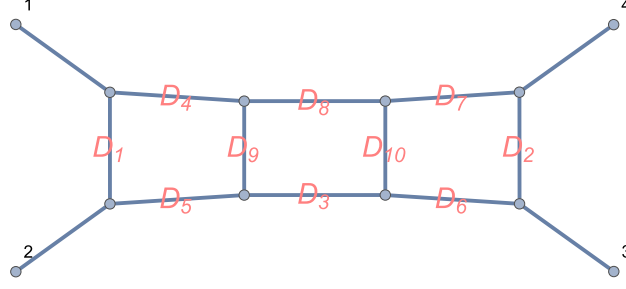


Figure 3.1: Three loop triple box diagram

diagram both have complicated multivariate residues. We discuss this direction in the rest of this chapter.

Note that, in a different context, [Hen15, CHH14, RT16, PT16] contour integrals like (3.5) from Feynman integrals are also important for determining the *canonical* MI basis [Hen13, Lee15, Mey16], for which the differential equation [Kot92, Kot91b, Kot91a, BDK94, Rem97, GR00] has a simple ϵ -form¹.

3.1.1 A multivariate residue example

Consider the $4D$ three-loop massless triple box diagram (Figure. 3.1). There are 10 inverse propagators,

$$\begin{aligned} D_1 &= l_1^2, & D_2 &= l_2^2, & D_3 &= l_3^2, & D_4 &= (l_1 + k_1)^2, \\ D_5 &= (l_1 - k_2)^2, & D_6 &= (l_2 + k_3)^2, & D_7 &= (l_2 - k_4)^2, & D_8 &= (l_3 + k_1 + k_2)^2, \\ D_9 &= (l_1 - l_3 - k_2)^2, & D_{10} &= (l_3 - l_2 - k_3)^2, \end{aligned} \quad (3.17)$$

with $k_1^2 = k_2^2 = k_3^2 = k_4^2 = 0$. We parameterize loop momenta with the spinor helicity formalism [Dix96],

$$\begin{aligned} \ell_1(\alpha_1, \dots, \alpha_4) &= \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 \frac{\langle 23 \rangle}{\langle 13 \rangle} 1\tilde{2} + \alpha_4 \frac{\langle 13 \rangle}{\langle 23 \rangle} 2\tilde{1}, \\ \ell_2(\beta_1, \dots, \beta_4) &= \beta_1 k_3 + \beta_2 k_4 + \beta_3 \frac{\langle 14 \rangle}{\langle 13 \rangle} 3\tilde{4} + \beta_4 \frac{\langle 13 \rangle}{\langle 14 \rangle} 4\tilde{3}, \\ \ell_3(\gamma_1, \dots, \gamma_4) &= \gamma_1 k_2 + \gamma_2 k_3 + \gamma_3 \frac{\langle 34 \rangle}{\langle 24 \rangle} 2\tilde{3} + \gamma_4 \frac{\langle 24 \rangle}{\langle 34 \rangle} 3\tilde{2}, \end{aligned} \quad (3.18)$$

The cut solution for $D_1 = D_2 = \dots = D_{10} = 0$ can be found by primary decomposition [BFZ12a, SZ13],

$$I = I_1 \cap \dots \cap I_{14}. \quad (3.19)$$

¹See [ABB⁺16] for the algorithm of solving differential equations without choosing a special basis, in the univariate case.

There are 14 independent solutions, each of which can be parameterized rationally. For example, on $V_1 = \mathcal{Z}(I_1)$ the triple box Feynman integral with numerator $N(l_1, l_2, l_3)$ becomes a contour integral,

$$\frac{1}{(2\pi i)^2 t^2 s^8} \oint \frac{dz_1 \wedge dz_2 N(l_1, l_2, l_3)|_{V_1}}{(1+z_1)(1+z_2)(1+z_1 - \chi z_2)}, \quad (3.20)$$

where the denominators come from the Jacobian of evaluating holomorphic delta functions in D_1, \dots, D_{10} . z_1, z_2 are free variables parametrizing this solution. The difficulty is that on this cut solution, loop momenta l_i are not polynomials in z_1 and z_2 , but rational functions in z_1 and z_2 [SZ13]. Hence we get contour integrals like,

$$\frac{1}{(2\pi i)^2} \oint \frac{dz_1 \wedge dz_2 P(z_1, z_2)}{(1+z_1)(1+z_2)(1+z_1 - \frac{t}{s} z_2)}. \quad (3.21)$$

where $P(z_1, z_2)$ is a polynomial in z_1 and z_2 . $(z_1, z_2) \rightarrow (-1, 0)$ is a multivariate residue. Note that at this point, 3 factors in the denominator vanish,

$$1+z_1, \quad z_2 \quad 1+z_1 - \frac{t}{s} z_2. \quad (3.22)$$

Hence, the Jacobian of denominators must be vanishing at $(-1, 0)$, so the residue cannot be calculated by inverse Jacobian (Cauchy's theorem). Note that we cannot directly use polynomial division to simplify the integrand, since $I = \langle 1+z_1, z_2, 1+z_1 - \frac{t}{s} z_2 \rangle = \langle 1+z_1, z_2 \rangle \neq \langle 1 \rangle$. So Hilbert's weak Nullstellensatz (Theorem 2.6) cannot be used here to reduce the number of denominators.

Difficult multivariate residues also arise from the maximal cut of integrals with doubled propagators from two-loop integrals. In the rest of this chapter, we use algebraic geometry techniques to compute these residues efficiently.

3.2 Basic facts of several complex variables

3.2.1 Multivariate holomorphic functions

We first review some properties of several complex variables [GH94, Hör90, Sch05].

Definition 3.1. *Complex variables for \mathbb{C}^n are $z_i = x_i + iy_i$ and the basis for the tangent space is*

$$\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right). \quad (3.23)$$

For a point $\xi = (\xi_1, \dots, \xi_n)$ in \mathbb{C}^n , the (open) polydisc with radius r is

$$\Delta(\xi, r) = \{(z_1, \dots, z_n) \mid |z_i - \xi_i| < r, \ i = 1, \dots, n\}. \quad (3.24)$$

Definition 3.2. *A differentiable function f on U , an open set of \mathbb{C}^n , is holomorphic if,*

$$\frac{\partial f}{\partial \bar{z}_i} = 0, \quad i = 1, \dots, n. \quad (3.25)$$

Theorem 3.3 (Cauchy's formula). *Let f a function holomorphic in $\Delta(\xi, r)$ and continuous on $\bar{\Delta}(\xi, r)$. Then for $z \in \Delta(\xi, r)$,*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|w_1 - \xi_1|=r} \cdots \int_{|w_n - \xi_n|=r} \frac{f(w_1, \dots, w_n) dw_1 \dots dw_n}{(w_1 - z_1) \dots (w_n - z_n)}. \quad (3.26)$$

Proof. Apply one-variable Cauchy's formula n times [Hör90]. \square

From the Taylor expansion of $1/(w_i - z_i)$ in $(z_i - \xi_i)$, $f(z)$ has a multivariate Taylor expansion in $\bar{\Delta}(\xi, r)$. Hence like in the univariate case, a holomorphic function is an analytic function. Similarly, for two holomorphic functions f and g on a connected open set $U \subset \mathbb{C}^n$, if $f = g$ on an open subset of U then $f = g$ on U .

However, the pole structure of a multivariate function is very different from that in the univariate case.

Theorem 3.4 (Hartog's extension). *Let U be an open set of \mathbb{C}^n , $n > 1$. Let K be a compact subset of U and $U - K$ be connected. Then any holomorphic function on $U - K$ extends to a holomorphic function of U .*

Proof. See Hörmander [Hör90, Chapter 2]. \square

Example 3.5. Consider $n = 2$, U is the polydisc $\Delta(0, r)$ and $K = O = \{(0, 0)\}$ in Theorem 3.4. Suppose that $f(z_1, z_2)$ is holomorphic in $U - K$. Define the function

$$g(z_1, z_2) = \frac{1}{2\pi i} \int_{|w_2|=r'} \frac{f(z_1, w_2) dw_2}{w_2 - z_2}, \quad (3.27)$$

where $0 < r' < r$. Clearly g is well defined in the smaller polydisc $\Delta(0, r')$. g is holomorphic in both z_1 and z_2 . If $z_1 \neq 0$, then by the univariate Cauchy's formula, $g(z_1, z_2) = f(z_1, z_2)$. $f = g$ in $\Delta(0, r') \cap \{z | z_1 \neq 0\}$, hence $f = g$ in $\Delta(0, r') - O$. Define a new function

$$F(z) = \begin{cases} g(z) & z \in \Delta(0, r') \\ f(z) & z \notin \Delta(0, r') \text{ but } z \in \Delta(0, r) \end{cases} \quad (3.28)$$

Clearly F is holomorphic in U and $F = f$ in $U - K$, so F is the extension.

Hartog's extension means the pole of a multivariate holomorphic f has complicated structure, say, cannot be a point. It also implies that we should not study the space of holomorphic functions on an open set like $U - K$ in Theorem 3.4, since these functions can always be extended.

Laurent expansion of a multivariate holomorphic function is also subtle.

Definition 3.6. A subset Ω of \mathbb{C}^n is called a Reinhardt domain, if Ω is open, connected and for any $(z_1, \dots, z_n) \in \Omega$, $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in \Omega$, $\forall \theta_1 \in \mathbb{R}, \dots, \theta_n \in \mathbb{R}$. This is a generalization of an annulus on the complex plane.

Proposition 3.7. Let f be a holomorphic function on a Reinhardt domain Ω . Then there exists a Laurent series,

$$\sum_{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n} c_{\alpha_1 \dots \alpha_n} z_1^{\alpha_1} \dots z_n^{\alpha_n}, \quad (3.29)$$

which is uniformly convergent to f on any compact subset of Ω .

Proof. See Scheidemann [Sch05]. □

A multivariate function f may be defined over a domain which is not a Reinhardt domain. For a simple example, the function $f = 1/(z_1 - z_2)$ is defined on $U = \{(z_1, z_2) | z_1 \neq z_2, (z_1, z_2) \in \mathbb{C}^2\}$, where U is not a Reinhardt domain. It is hard to define the Laurent series for f in U . Instead, consider $\Omega = \{(z_1, z_2) | |z_1| > |z_2|, (z_1, z_2) \in \mathbb{C}^2\}$. Then Ω is a Reinhardt domain, and on Ω ,

$$\frac{1}{z_1 - z_2} = \sum_{n=0}^{\infty} z_2^n z_1^{-n-1}, \quad (z_1, z_2) \in \Omega \quad (3.30)$$

This Laurent series does not converge outside Ω .

We turn to complex manifolds.

Definition 3.8. A complex manifold M is a differentiable manifold, with an open cover $\{U_\alpha\}$ and coordinate maps $\phi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$, such that all $\phi_\alpha \phi_\beta^{-1}$'s components are holomorphic on $\phi_\beta(U_\alpha \cap U_\beta)$ for $U_\alpha \cap U_\beta \neq \emptyset$.

Example 3.9 (Complex projective space). Define \mathbb{CP}^n as the quotient space $\mathbb{C}^{n+1} - \{0, \dots, 0\}$ over

$$(Z_0, \dots, Z_n) \sim (\lambda Z_0, \dots, \lambda Z_n), \quad \lambda \in \mathbb{C}^* \quad (3.31)$$

The equivalence class of (Z_0, \dots, Z_n) in \mathbb{CP}^n is denoted as $[Z_0, \dots, Z_n]$, which is the homogeneous coordinate. Define an open cover of \mathbb{CP}^n , $\{U_0, U_1, \dots, U_n\}$, where

$$U_i = \{[Z_0, \dots, Z_n] | Z_i \neq 0\}, \quad i = 0, \dots, n. \quad (3.32)$$

For each U_i , the coordinate map $\phi_i : U_i \rightarrow \mathbb{C}^n$ is

$$\phi_i([Z_0, \dots, Z_n]) = \left(\frac{Z_0}{Z_i}, \dots, \widehat{\frac{Z_i}{Z_i}}, \dots, \frac{Z_n}{Z_i} \right) \equiv (z_0^{(i)}, \dots, \widehat{z_i^{(i)}}, \dots, z_n^{(i)}). \quad (3.33)$$

Hence, for $i < j$,

$$\phi_i \phi_j^{-1}(z_0^{(j)}, \dots, \widehat{z_j^{(j)}} \dots z_n^{(j)}) = \left(\frac{z_0^{(j)}}{z_i^{(j)}}, \dots, \widehat{\frac{z_i^{(j)}}{z_i^{(j)}}}, \dots, \frac{1}{z_i^{(j)}}, \dots, \frac{z_n^{(j)}}{z_i^{(j)}} \right). \quad (3.34)$$

Since on $\phi_j(U_i \cap U_j)$, $z_i^{(j)} \neq 0$, the transformation (3.34) is holomorphic. Hence \mathbb{CP}^n is a compact complex space. In particular, we may identify U_0 as \mathbb{C}^n .

For a homogeneous polynomial $F(Z_0, \dots, Z_n)$, the equation $F(Z_0, \dots, Z_n) = 0$ is well defined, since the rescaling (3.31) does not affect the value 0.

Like the real manifold case, we can also study sub-manifolds of a manifold. In particular, the codimension-1 case is very important for our discussion of residues in this chapter.

Definition 3.10. An analytic hypersurface V of a complex manifold M is a subset of M such that $\forall p \in V$, there exists a neighborhood of p in M , such that locally V is the set of zeros of a holomorphic function f , defined in this neighborhood.

Like in the algebraic variety case (Theorem 2.35), any analytic hypersurface uniquely decomposes as the union of irreducible analytic hypersurfaces. [GH94].

Definition 3.11. For a complex manifold M , a divisor D is a locally finite formal linear combination

$$D = \sum_i c_i V_i, \quad (3.35)$$

where each V_i is an irreducible analytic hypersurface in M .

3.2.2 Multivariate residues

Recall that in the univariate case, the residue of a meromorphic function $h(z)/f(z)$ at the point ξ , is defined as

$$\text{Res}_\xi \left(\frac{h(z)}{f(z)} dz \right) = \frac{1}{2\pi i} \oint_{|z-\xi|=\epsilon} \frac{h(z)dz}{f(z)}. \quad (3.36)$$

where f and h are holomorphic near ξ .

To define a multivariate residue in \mathbb{C}^n , we need n vanishing denominators f_1, \dots, f_n such that $f_1(z) = \dots = f_n(z) = 0$ defines isolated points.

Definition 3.12. Let U be a ball in \mathbb{C}^n centered at ξ , i.e. $\|z - \xi\| < \epsilon$ for $z \in U$. Assume that $f_1(z), \dots, f_n(z)$ are holomorphic function in U , and have only one isolated common zero, ξ in U . Let $h(z)$ be a holomorphic function in a neighborhood of \bar{U} . Then for the differential form

$$\omega = \frac{h(z)dz_1 \wedge \dots \wedge dz_n}{f_1(z) \cdots f_n(z)}, \quad (3.37)$$

the (Grothendieck) residue [GH94] at ξ is defined to be

$$\text{Res}_{\{f_1, \dots, f_n\}, \xi}(\omega) = \left(\frac{1}{2\pi i} \right)^n \oint_\Gamma \frac{h(z)dz_1 \wedge \dots \wedge dz_n}{f_1(z) \cdots f_n(z)}, \quad (3.38)$$

where the contour Γ is defined by the real n -cycle $\Gamma = \{z : z \in U, |f_i(z)| = \epsilon\}$ with the orientation specified by $d(\arg f_1) \wedge \dots \wedge d(\arg f_n)$.

Note that Γ in this definition ensures that f_i 's are nonzero for this contour integral. A naive contour choice $\Gamma' = \{z : z \in U, |z_i - \xi_i| = \epsilon, \forall i\}$ in general does not work. For instance,

$$\frac{1}{(2\pi i)^2} \oint_{\Gamma'} \frac{dz_1 \wedge dz_2}{(z_1 + z_2)(z_1 - z_2)}, \quad (3.39)$$

with $\Gamma' = \{z : z \in U, |z_1| = \epsilon, |z_2| = \epsilon\}$ is ill-defined. On this contour, both $(z_1 + z_2)$ and $(z_1 - z_2)$ have zeros.

Note that if we permute functions $\{f_1, \dots, f_n\}$, the differential form is invariant but the contour orientation will be reversed if the permutation is odd. This is a new feature of multivariate residue, hence in Definition 3.12, we keep $\{f_1, \dots, f_n\}$ in the subscript.

Clearly, if $f_1(z) = f_1(z_1), \dots, f_n(z) = f_n(z_n)$, then

$$\text{Res}_{\{f_1, \dots, f_n\}, \xi}(\omega) = \left(\frac{1}{2\pi i}\right)^n \oint \frac{dz_1}{f_{z_1}} \oint \frac{dz_2}{f_{z_2}} \dots \oint \frac{h(z)dz_n}{f_{z_n}}, \quad (3.40)$$

i.e., the multivariate residue becomes iterated univariate residues.

Definition 3.13. In Definition 3.12, if the Jacobian of f_1, \dots, f_n in z_1, \dots, z_n at ξ is nonzero, we call this residue non-degenerate. Otherwise it is called degenerate.

Proposition 3.14 (Cauchy). If the residue in Definition 3.12 is non-degenerate, then

$$\text{Res}_{\{f_1, \dots, f_n\}, \xi}(\omega) = \frac{h(\xi)}{J(\xi)}. \quad (3.41)$$

where $J(\xi)$ is the Jacobian of f_1, \dots, f_n in z_1, \dots, z_n at ξ .

Proof. In this case, we can use implicit function theorem to treat f_i 's as coordinates and compute the residue directly [GH94]. \square

Proposition 3.15. If h in Definition 3.12 satisfies,

$$h(z) = q_1(z)f_1(z) + \dots + q_n(z)f_n(z), \quad z \in U \quad (3.42)$$

where q_i 's are holomorphic functions in U . Then

$$\text{Res}_{\{f_1, \dots, f_n\}, \xi}(\omega) = 0. \quad (3.43)$$

Proof. This is from Stokes' theorem [GH94]. \square

In general a multivariate residue is not of a form like (3.40) or non-degenerate. Unlike the univariate case, Laurent expansion, even if it is defined in a subset, in general does not help the evaluation of multivariate residues. Hence we need a sophisticated method to compute residues, like (3.21).

Theorem 3.16 (Global residue). Let M be a compact complex manifold, and $D_1 \dots D_n$ be divisors of M , such that $D_1 \cap \dots \cap D_n = S$ is a finite set. If ω is a holomorphic n -form defined in $M - D_1 \cup \dots \cup D_n$ whose polar divisor is $D = D_1 + \dots + D_n$, then

$$\sum_{\xi \in S} \text{Res}_{\{D_1, \dots, D_n\}, \xi}(\omega) = 0. \quad (3.44)$$

Proof. This is from Stokes' theorem for a complex manifold. See Griffiths and Harris [GH94]. \square

Note that to consider global residue theorem, we need a compact complex manifold but \mathbb{C}^n is not. So residues on a complex manifold have to be defined. ω has the polar divisor

$D = D_1 + \dots + D_n$ means, near a point $\xi \in S$, locally each D_i is a divisor of a holomorphic function f_i and ω has the local form (3.37). Again, the subscript $\{D_1, \dots, D_n\}$ indicates the ordering of denominators.

Example 3.17. Consider the meromorphic differential form in \mathbb{C}^n ,

$$\omega = \frac{dz_1 \wedge dz_2}{(z_1 + z_2)(z_1 - z_2)}. \quad (3.45)$$

Extend ω to a meromorphic differential form in \mathbb{CP}^2 (Example 3.9). Let $[Z_0, Z_1, Z_2]$ be the homogeneous coordinate. In the patch U_0 , define $z_1 = Z_1/Z_0$, $z_2 = Z_2/Z_0$. For the patch U_1 , let $u_0 = Z_0/Z_1$, $u_2 = Z_2/Z_1$. Then on $U_0 \cap U_1$,

$$z_1 = \frac{1}{u_0}, \quad z_2 = \frac{u_2}{u_0}. \quad (3.46)$$

After a change of variables, on $U_0 \cap U_1$,

$$\omega = \frac{du_0 \wedge du_2}{u_0(u_2 - 1)(u_2 + 1)}. \quad (3.47)$$

Similarly, for the patch U_2 , let $v_0 = Z_0/Z_2$, $v_1 = Z_1/Z_2$. On $U_0 \cap U_2$,

$$\omega = \frac{dv_0 \wedge dv_1}{v_0(v_1 - 1)(v_1 + 1)}. \quad (3.48)$$

Then in \mathbb{CP}^2 , ω is defined except on 3 irreducible hypersurfaces $V_1 = \{Z_0 = 0\}$, $V_2 = \{Z_1 + Z_2 = 0\}$ and $V_3 = \{Z_1 - Z_2 = 0\}$. To apply the global residue theorem, consider

$$D_1 = V_0 + V_1, \quad D_2 = V_2. \quad (3.49)$$

Then $D = D_1 + D_2$ is the polar divisor of ω . $D_1 \cap D_2 = \{P_1, P_2\}$, where $P_1 = [1, 0, 0]$ and $P_2 = [0, 1, 1]$. The global residue theorem reads,

$$\text{Res}_{\{D_1, D_2\}, P_1}(\omega) + \text{Res}_{\{D_1, D_2\}, P_2}(\omega) = 0. \quad (3.50)$$

Explicitly by (3.41),

$$\text{Res}_{\{D_1, D_2\}, P_1}(\omega) = -\frac{1}{2}, \quad \text{Res}_{\{D_1, D_2\}, P_2}(\omega) = \frac{1}{2}. \quad (3.51)$$

Note that if we consider a different set of divisors, say, $D'_1 = V_1$, $D'_2 = V_0 + V_2$, then $D'_1 \cap D'_2 = \{P_1, P_3\}$, where $P_3 = [0, 1, -1]$. So there is another relation $\text{Res}_{\{D'_1, D'_2\}, P_1}(\omega) + \text{Res}_{\{D'_1, D'_2\}, P_3}(\omega) = 0$, and,

$$\text{Res}_{\{D'_1, D'_2\}, P_3}(\omega) = \frac{1}{2}. \quad (3.52)$$

We see that for a multivariate case, there can be several global residues relations for one meromorphic form.

3.3 Multivariate residues via computational algebraic geometry

There are several algorithms for calculating multivariate residues in algebraic geometry. We mainly use two methods, the *transformation law* and the *Bezoutian*.

3.3.1 Transformation law

Theorem 3.18. *For the residue in Definition 3.12, and $g_i = \sum_j a_{ij} f_j$, where a_{ij} are locally holomorphic functions near ξ . We have*

$$\text{Res}_{\{f_1, \dots, f_n\}, \xi}(\omega) = \text{Res}_{\{g_1, \dots, g_n\}, \xi}(\det A \omega) \quad (3.53)$$

where A is the matrix (a_{ij}) .

Proof. See Griffiths and Harris [GH94]. □

Note that this is a transformation of denominators, not the complex variables. In particular, if f_1, \dots, f_n are polynomials, we can calculate the Gröbner basis for $I = \langle f_1, \dots, f_n \rangle$ in *lex* to get a set of polynomial g_i 's, such that each g_i is univariate. $g_i(z) = g_i(z_i)$ (Theorem 2.26). Then the r.h.s of (3.53) can be calculated as univariate residues.

Example 3.19. *Consider the residue of*

$$\omega = \frac{dx \wedge dy}{f_1 f_2} \quad (3.54)$$

at $(0, 0)$, where $f_1 = ay^3 + x^2 + y^2$, $f_2 = x^3 + xy - y^2$. This is a degenerate residue. By Gröbner basis computations,

$$A = \begin{pmatrix} \frac{-2ax^2 + ax - ayx - ay + 1}{a^2 y^5 - 2ay^3 - ax^2 y^2 + xy + y + x^2} & \frac{ax^4 - ayx^2 + ay^2 x - x + ay^2 - y}{a^3} \\ \frac{a^2 y^5 - 2ay^3 - ax^2 y^2 + xy + y + x^2}{a^3} & \frac{axy^2 - y - x}{a^3} \end{pmatrix}, \quad (3.55)$$

and,

$$\{g_1, g_2\} = \left\{ \frac{x^2(a^2 x^5 - 3ax^2 - ax - 1)}{a^2}, \frac{y^3(a^3 y^5 - 2a^2 y^3 + ay + 1)}{a^3} \right\}. \quad (3.56)$$

Note that g_1, g_2 are univariate polynomials. Hence by (3.53),

$$\text{Res}_{\{f_1, f_2\}, (0,0)}(\omega) = a(1 - a). \quad (3.57)$$

Example 3.20. *Consider the 4D triple box's maximal cut (3.21), near $z_1 = -1$ and $z_2 = 0$,*

$$\omega = \frac{dz_1 \wedge dz_2 P(z_1, z_2)}{(1 + z_1)(1 + z_2)(1 + z_1 - \frac{t}{s} z_2) z_2}. \quad (3.58)$$

Define $V_1 = \{1 + z_1 = 0\}$, $V_2 = \{z_2 = 0\}$ and $V_3 = \{1 + z_1 - \chi z_2\}$, which are irreducible

hypersurfaces. So locally the polar divisor of ω is,

$$D = V_1 + V_2 + V_3. \quad (3.59)$$

To define multivariate residues, we may consider two divisors $D_1 = V_1 + V_2$ and $D_2 = V_3$. This corresponds to the denominator definitions, $f_1 = (1 + z_1)z_2$ and $f_2 = (1 + z_1 - t/sz_2)$. Using (3.53) to change denominators, we find that, for example if $P = 1$,

$$\text{Res}_{\{f_1, f_2\}, (0,0)}(\omega) = s/t. \quad (3.60)$$

Note that there are different ways to define the divisors for ω , for instance, $D'_1 = V_1 + V_3$ and $D'_2 = V_2$, i.e. $f'_1 = (1 + z_1)(1 + z_1 - \chi z_2)$ and $f'_2 = z_2$. Multivariate residue dependence on the definition of divisors, for example if $P = 1$,

$$\text{Res}_{\{f'_1, f'_2\}, (0,0)}(\omega) = 0 \neq \text{Res}_{\{f_1, f_2\}, (0,0)}(\omega). \quad (3.61)$$

Hence we need to consider all possible divisor definitions.

We calculated all 64 residues from the maximal unitarity cut of a three-loop triple box diagram [SZ13], by Cauchy's theorem and transformation law. Then the contours weights are determined by spurious integrals and IBPs. We used contour weights to derive the triple box master integrals part of 4-gluon 3-loop pure-Yang-Mills amplitude, which agrees with that from integrand reduction method [BFZ12a].

For integrals with doubled propagators, we can also use the transformation law to compute residues for contour integrals [SZ14b, SZ14a].

Remark.

1. Usually, Gröbner basis computation in lex is heavy. It is better to first compute Gröbner basis in grevlex order, $G(I) = \{F_1, \dots, F_k\}$ and find the relations $F_i = b_{ij}f_j$. Then compute Gröbner basis in a block order to get univariate polynomials $g_i(z_i)$. Divide $g_i(z_i)$ towards $G(I)$ and use b_{ij} 's, we get the matrix A .
2. This method also works if f_1, \dots, f_n are holomorphic functions but not polynomials. Replace f_i 's by their Taylor series, we can apply the Gröbner basis computation.

3.3.2 Bezoutian

Multivariate residue computation via Gröbner basis, may be quite heavy since the transformation matrix A may contain high-degree polynomials. The Bezoutian method provide a different approach.

Definition 3.21. With the convention of Definition 3.12, for $\xi \in \mathbb{C}^n$, define the local symmetric form, for locally holomorphic functions N_1 and N_2 ,

$$\langle N_1, N_2 \rangle_\xi \equiv \text{Res}_{\{f_1, \dots, f_n\}, \xi} \left(\frac{N_1 N_2 dz_1 \wedge \dots \wedge dz_n}{f_1 \dots f_n} \right). \quad (3.62)$$

If f_1, \dots, f_n , N_1, N_2 are globally holomorphic in \mathbb{C}^n and $\mathcal{Z}(\{f_1, \dots, f_n\})$ is a finite set, then the global symmetric form is

$$\langle N_1, N_2 \rangle \equiv \sum_{\xi \in \mathcal{Z}(\{f_1, \dots, f_n\})} \text{Res}_{\{f_1, \dots, f_n\}, \xi} \left(\frac{N_1 N_2 dz_1 \wedge \dots \wedge dz_n}{f_1 \dots f_n} \right). \quad (3.63)$$

For the rest of the discussion, we assume f_1, \dots, f_n , N_1, N_2 are polynomials. In the previous Chapter, we used the ring $R = \mathbb{F}[x_1, \dots, x_n]$ and ideals to study algebraic varieties. Here to discuss local properties of a variety, we need the concept of the local ring.

Definition 3.22. Consider $R = \mathbb{C}[x_1, \dots, x_n]$, for a point $\xi \in \mathbb{C}^n$, R_ξ is the set of rational functions,

$$R_\xi \equiv \left\{ \frac{f(z)}{g(z)} \mid g(\xi) \neq 0, \quad f, g \in R \right\}. \quad (3.64)$$

For an ideal I in R , we denote I_ξ as the ideal in R_ξ generated by I . If $\xi \in \mathcal{Z}(I)$, and $\dim_{\mathbb{C}} R_\xi / I_\xi < \infty$, we define the multiplicity of I at ξ as $\dim_{\mathbb{C}} R_\xi / I_\xi$.

Let I be $\langle f_1, \dots, f_n \rangle$. From Proposition 3.15, it is clear that \langle, \rangle is defined in R/I and \langle, \rangle_ξ is defined in R_ξ / I_ξ , because any polynomial in the ideal I or the localized ideal I_ξ must yield zero residue.

Theorem 3.23 (Local and Global Dualities). *Let $I = \langle f_1, \dots, f_n \rangle$ be an ideal in $\mathbb{C}[x_1, \dots, x_n]$, $\mathcal{Z}(I)$ is a finite set. Then \langle, \rangle is non-degenerate in R/I and \langle, \rangle_ξ is non-degenerate in R_ξ / I_ξ .*

Proof. See Griffiths and Harris [GH94], Dickenstein et al. [DE10]. □

Non-degeneracy of \langle, \rangle implies that given a linear basis $\{e_1, \dots, e_k\}$ of R/I , there is a dual basis $\{\Delta_1, \dots, \Delta_n\}$, such that,

$$\langle e_i, \Delta_j \rangle = \delta_{ij}. \quad (3.65)$$

If these two bases are explicitly found, then we can compute any $\langle N_1, N_2 \rangle$. In particular, the sum of residues (in affine space) of $\omega = N dz_1 \wedge \dots \wedge dz_n / (f_1 \dots f_n)$ is obtained algebraically,

$$\sum_{\xi \in \mathcal{Z}(I)} \text{Res}_{\{f_1, \dots, f_n\}, \xi}(\omega) = \langle N, 1 \rangle = \left\langle \sum_{i=1}^k c_i e_i, \sum_{j=1}^k \mu_j \Delta_j \right\rangle = \sum_{i=1}^k c_i \mu_i, \quad (3.66)$$

where in the second equality, we expand $N = \sum_i c_i e_i$ and $1 = \sum_i \mu_i \Delta_i$. c_i 's and Δ_i 's are complex numbers.

Explicitly, $\{e_i\}$'s are found by using Gröbner basis of I in *grevlex*, $G(I)$. They are monomials which are not divisible by any leading term in $G(I)$. The dual basis can be found via the Bezoutian matrix [DE10]. First, calculate the Bezoutian matrix $B = (b_{ij})$,

$$b_{ij} \equiv \frac{f_i(y_1, \dots, y_{j-1}, z_j, \dots, z_n)}{z_j - y_j} - \frac{f_i(y_1, \dots, y_j, z_{j+1}, \dots, z_n)}{z_j - y_j}, \quad (3.67)$$

where y_i 's are auxiliary variables. Let \tilde{I} be the ideal in $\mathbb{C}[y_1, \dots, y_n]$ which is obtained from I after the replacement $z_1 \rightarrow y_1, \dots, z_n \rightarrow y_n$.

Then we divide the determinant $\det B$ over the double copy of the Gröbner bases, $G(I) \otimes G(\tilde{I})$. The remainder can be expanded as,

$$\sum_{i=1}^k \Delta_i(y) e_i(z), \quad (3.68)$$

here the $\Delta_i(y)$'s, after the backwards replacement $y_1 \rightarrow z_1, \dots, y_n \rightarrow z_n$ become the elements of the dual basis [DE10].

Example 3.24. Let $f_1 = z_1 + 9z_2 + 14z_3 + 6$, $f_2 = 11z_2z_1 + 12z_3z_1 + 3z_1 + 4z_2 + 16z_2z_3 + 14z_3$ and $f_3 = 2z_1z_2 + 15z_1z_3z_2 + 5z_3z_2 + 8z_1z_3$ be polynomials in $\mathbb{C}[z_1, z_2, z_3]$. Define

$$\omega = \frac{z_1^3 dz_1 \wedge dz_2 \wedge dz_3}{f_1 f_2 f_3}. \quad (3.69)$$

The Bezoutian determinant in z_1, z_2, z_3 and auxiliary variables y_1, y_2, y_3 is,

$$\begin{aligned} \det B = & -180y_1^2z_3 + 2520y_1z_3^2 - 1485y_2y_1z_2 - 576y_1z_2 - 1620y_2y_1z_3 \\ & + 1620y_1z_2z_3 - 408y_1z_3 - 207y_2z_2 + 612y_2z_3 + 2160y_2z_2z_3 + 165y_2y_1^2 + 64y_1^2 - 322y_2y_1 \\ & - 128y_1 - 115y_2 - 3360z_2z_3^2 - 952z_3^2 + 140z_2 + 1372z_2z_3 + 700z_3. \end{aligned} \quad (3.70)$$

Let $I = \langle f_1, f_2, f_3 \rangle$. Divide $\det B$ towards $G(I) \otimes G(\tilde{I})$, and we get the basis $\{e_i\}$,

$$e_1 = z_3^3, \quad e_2 = z_2z_3, \quad e_3 = z_3^2, \quad e_4 = z_2, \quad e_5 = z_3, \quad e_6 = 1, \quad (3.71)$$

and the dual basis $\{\Delta_i\}$,

$$\begin{aligned} \Delta_1 &= \frac{141120}{23}, \quad \Delta_2 = 2(-12420z_2 - 22680z_3 - \frac{203652}{23}), \\ \Delta_3 &= -22680z_2 - 35280z_3 - \frac{335832}{23}, \\ \Delta_4 &= 2(-22680z_3^2 - 12420z_2z_3 - 5436z_3 + 1872z_2 + 1278), \\ \Delta_5 &= -35280z_3^2 - 22680z_2z_3 - 24528z_3 - 5436z_2 - \frac{79884}{23}, \\ \Delta_6 &= \frac{141120z_3^3}{23} - \frac{335832z_3^2}{23} - \frac{203652z_2z_3}{23} - \frac{79884z_3}{23} + 1278z_2 + \frac{21282}{23}. \end{aligned} \quad (3.72)$$

From the dual basis, we find the linear relation,

$$1 = \frac{23}{141120} \Delta_1. \quad (3.73)$$

By polynomial division, we find

$$z_1^3 = \frac{1568}{11} e_1 + c_2 e_2 + \dots c_6 e_6 \mod I. \quad (3.74)$$

Hence the sum of residues,

$$\begin{aligned} \sum_{\xi \in \mathcal{Z}(I)} \text{Res}_{\{f_1, f_2, f_3\}, \xi}(\omega) &= \langle z_1^3, 1 \rangle \\ &= \frac{23}{141120} \langle \frac{1568}{11} e_1 + c_2 e_2 + \dots + c_6 e_6, \Delta_1 \rangle = \frac{23}{990}. \end{aligned} \quad (3.75)$$

Note that all points in $\mathcal{Z}(I)$ and all local residues are irrational, but the sum is rational.

This example is from the CHY formalism of scattering equation [CHY14a, CHY14b, CHY14c, CHY15a, CHY15b] for 6-point tree amplitudes. In the CHY formalism, scattering amplitudes are expressed as the sum of residues of CHY integrand. Here we calculate the amplitude without solving the scattering equations [SZ16]. See alternative algebraic approaches in [BBBBD15a, BBBBD15b, HRFH15].

Remark.

1. Note that by this method, we get the sum of residues (in affine space) purely by Gröbner basis and matrix determinant computations. It is not needed to consider algebraic extension or explicit solutions of $f_1 = \dots = f_n = 0$.
2. The Bezoutian matrix is just an $n \times n$ matrix, i.e., the size of matrix is independent of the dimension $\dim_{\mathbb{C}} R/I$. Hence it is an efficient method for computing the sum of residues.
3. If f_i 's coefficients are parameters, this method proves that the sum of residues is a rational function of these parameters.
4. In some cases, the sum of residues can also be evaluated by the global residue theorem (GRT). However, in general, there are many poles at infinity so the GRT computation can be messy.

We can also use the Bezoutian matrix to find local residues. One approach is *partition of unity* for an affine variety: For each $\xi \in \mathcal{Z}(I)$, we can find a polynomial s_{ξ} [CLO98], such that,

$$\begin{aligned} \sum_{\xi \in \mathcal{Z}(I)} s_{\xi} &= 1 \pmod{I}, \quad s_{\xi}^2 = s_{\xi} \pmod{I}, \\ s_{\xi_i} s_{\xi_j} &= 0 \pmod{I}, \quad \text{if } i \neq j. \end{aligned} \quad (3.76)$$

Then the individual residue is extracted from the sum of residues,

$$\text{Res}_{\{f_1, \dots, f_n\}, \xi}(\omega) = \sum_{u \in \mathcal{Z}(I)} \text{Res}_{\{f_1, \dots, f_n\}, u}(s_{\xi} \omega), \quad (3.77)$$

where the r.h.s is again obtained by Bezoutian matrix computation [DE10].

3.4 Exercises

Exercise 3.1. Consider the maximal unitarity cut of $D = 2$ massless sunset diagram with $k_1^2 = M^2$ and inverse propagators,

$$D_1 = l_1^2, \quad D_2 = l_2^2, \quad D_3 = (l_1 + l_2 - k_1)^2. \quad (3.78)$$

1. Define an auxiliary vector ω , $k_1 \cdot \omega = 0$, $\omega^2 = -M^2$. Let $e_1 = (k_1 + \omega)/2$ and $e_2 = k_1 - \omega$. Parameterize the loop momenta as,

$$l_1 = a_1 e_1 + a_2 e_2, \quad l_2 = b_1 e_1 + b_2 e_2. \quad (3.79)$$

Rewrite the D_i 's as polynomials in a_1, a_2, b_1, b_2 . Define $I = \langle D_1, D_2, D_3 \rangle$, use SINGULAR or MACAULAY2 to find independent solutions via primary decomposition,

$$I = I_1 \cap \dots \cap I_m. \quad (3.80)$$

2. Formally define,

$$I[s_1, s_2, s_3; N] = \int \frac{d^2 l_1}{(2\pi)^2} \frac{d^2 l_2}{(2\pi)^2} \frac{N}{D_1^{s_1} D_2^{s_2} D_3^{s_3}}. \quad (3.81)$$

Consider the maximal cut of the scalar integral $I[1, 1, 1; 1]$ on each of the cut solutions $\mathcal{Z}(I_i)$. From the resulting contour integrals, determine all the poles on maximal cut. How many of them are redundant?

3. Denote independent poles as $\{P_1, \dots, P_k\}$ and denote $I[s_1, s_2, s_3; N]_{P_i}$ as the residue of its corresponding contour integral at P_i . Compute $I[1, 1, 1; 1]_{P_i}$ for all P_i .

4. Denote

$$I[s_1, s_2, s_3; N]_{\text{cut}} = \sum_i^k w_i I[s_1, s_2, s_3; N]_{P_i}, \quad (3.82)$$

where the w_i 's are weights of contours. Require that $I[1, 1, 1; N]_{\text{cut}} = 0$ for spurious terms N ,

$$l_1 \cdot \omega, \quad l_2 \cdot \omega, \quad (l_1 \cdot \omega)(l_2 \cdot k_1), \quad l_2 \cdot k_1 - l_1 \cdot k_1, \quad (l_2 \cdot k_1)^2 - (l_1 \cdot k_1)^2. \quad (3.83)$$

What are the linear constraints on the w_j 's?

5. Determine the ratio, $I[2, 1, 1; 1]_{\text{cut}}/I[1, 1, 1; 1]_{\text{cut}}$. Derive the on-shell integral relation (by determining c)

$$I[2, 1, 1; 1] = c I[1, 1, 1; 1] + (\text{simpler integrals}). \quad (3.84)$$

Similarly, determine c' in

$$I[3, 1, 1; 1] = c' I[1, 1, 1; 1] + (\text{simpler integrals}). \quad (3.85)$$

Exercise 3.2. Consider the meromorphic form,

$$\omega = \frac{z_1 dz_1 \wedge dz_2}{(z_1 + z_2)(z_1 - z_2 + z_1 z_2)}. \quad (3.86)$$

Extend ω on a meromorphic form in \mathbb{CP}^2 . Find all residues of ω in \mathbb{CP}^2 and verify the global residue theorem explicitly.

Exercise 3.3. Consider the meromorphic form,

$$\omega = \frac{N(z_1, z_2) dz_1 \wedge dz_2}{(z_1 + az_2)(z_1^3 + z_2^2 + bz_1 z_2)}. \quad (3.87)$$

1. Use the transformation law and Gröbner basis computation in MAPLE or MACAULAY2, to compute the residue at $(0, 0)$ with $N(z_1, z_2) = 1$ and $N(z_1, z_2) = z_1$.
2. Without computation, argue that if $N(z_1, z_2) = z_1^2$ then the residue at $(0, 0)$ is zero by Proposition 3.15.

Exercise 3.4. Consider the meromorphic form,

$$\omega = \frac{N(z_1, z_2) dz_1 \wedge dz_2}{(z_1 + z_2)(z_1 - z_2)(z_1^2 + z_2^2 + z_1)}. \quad (3.88)$$

Define $f_1 = (z_1 + z_2)$, $f_2 = (z_1 - z_2)$ and $f_3 = (z_1^2 + z_2^2 + z_1)$. Use the transformation law to compute,

$$\text{Res}_{\{f_1, f_2, f_3\}, (0,0)}(\omega), \quad \text{Res}_{\{f_1 f_2, f_3\}, (0,0)}(\omega). \quad (3.89)$$

Exercise 3.5 (Sum of residues). Consider $f_1 = z_1^2 + z_1 z_2 + az_2$, $f_2 = z_1^3 + z_2^2 + bz_1 z_2$ and $I = \langle f_1, f_2 \rangle$.

1. Using Gröbner basis in grevlex, determine the basis $\{e_i\}$ for $\mathbb{C}[z_1, z_2]/I$.
2. Using Bezoutain matrix, find the dual basis $\{\Delta_i\}$.
3. Compute the sum of residues in \mathbb{C}^n for

$$\omega = \frac{z_1 z_2^2 dz_1 \wedge dz_2}{f_1 f_2}. \quad (3.90)$$

4. Compute $\langle e_i, e_j \rangle$ for all elements in $\{e_i\}$. Define $s_{ij} = \langle e_i, e_j \rangle$ and check that $S = (s_{ij})$ is a symmetric non-degenerate matrix.

Chapter 4

Integration-by-parts Reduction and Syzygies

Integration-by-parts (IBP) identities [Tka81, CT81], arise from the vanishing integration of total derivatives. Combined with symmetry relations, IBPs reduce integrals to master integrals (MIs), i.e., the linearly independent integrals.

An L -loop D -dimensional ¹ IBP in general has the form,

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \sum_{j=1}^L \frac{\partial}{\partial l_j^\mu} \left(\frac{v_j^\mu}{D_1^{a_1} \cdots D_k^{a_k}} \right) = 0, \quad (4.1)$$

where the vectors components v_j^μ 's are polynomials in the internal and external momenta, the D_k 's denote inverse propagators, and the a_i 's are integers.

For many multi-loop scattering amplitudes, IBP reduction is a necessary step. After using unitarity and integrand reduction to obtain the integrand basis, we may carry out IBP reduction to get the minimal basis of integrals. For differential equations of Feynman integrals [Kot92, Kot91b, Kot91a, BDK94, Rem97, GR00], after differentiating of the master integrals, we get a large number of integrals in general. Then IBP reduction is required to convert them to a linear combination of MIs, so that the differential equation system is closed.

Multi-loop IBP reduction in general is very difficult. The difficulty comes from the large number of choices of v_i^μ in (4.1) : there are many IBP relations and integrals involved. After obtaining IBP relations, we need to apply linear reduction to find the independent set of IBPs. This process usually takes a lot of computing time and RAM. The current standard IBP generating algorithm is Laporta [Lap00, Lap01]. There are several publicly available implementations of automated IBP reduction: AIR [AL04], FIRE [Smi08, Smi15], Reduze [Stu10, vMS12], LiteRed [Lee12], along with private implementations. IBP computation can be sped up by using finite-field methods [vMS15, vMS16].

One sophisticated way to improve the IBP generating efficiency is to pick up suitable v_i^μ 's such that (4.1) contains no doubled propagator [GKK11]. Since from Feynman rules,

¹In general, we need to consider IBP in D -dimension. Otherwise for a specific integer-valued D , IBP relations may contain non-vanishing boundary terms.

usually we only have integrals without doubled propagators. Hence if we can work with integrals without doubled propagators during the whole IBP reduction procedure, the computation will be significantly simplified. Specifically, when $a_i = 1, \forall i = 1, \dots, k$ in (4.1), if

$$\sum_j \frac{\partial D_i}{\partial l_j^\mu} v_i^\mu = \beta_i D_i, \quad i = 1 \dots k, \quad (4.2)$$

where β_i is a polynomial in loop momenta, then all double-propagator integrals are removed from the IBP relation (4.1).

Note that (4.2) appears to be a linear equation system for v_i^μ 's and β_i . However, v_i^μ 's must be polynomials in loop momenta, otherwise the doubled propagators reappear. If we solve (4.2) by standard linear algebra method, then the solutions are in general rational functions which do not help the IBP reduction. To distinguish with linear equations, (4.2) is called a *syzygy* equation. It is not surprising that the form of (4.2) is closely related to S-polynomials and polynomial division (Definition 2.17 and Algorithm 2), so the syzygy can be solved by Gröbner basis.

It is convenient to consider IBPs in various integral representations, like Feynman parametric representation or Baikov representation [Bai96]. We believe that the syzygy approach (4.2) maximizes its power, when combined with Baikov representation and unitarity cuts. (See using syzygy approach in Feynman parametric representation [LP13, Lee14].) Baikov representation linearizes inverse propagators D_i 's so the syzygy equation becomes simpler. Furthermore, It is more efficient to compute IBPs with unitarity cuts, in a divide-and-conquer fashion, than to get complete IBPs at once.

In this chapter, we first introduce the Baikov representation and then review syzygy and the geometric meaning of (4.2). We will see that it defines *polynomial tangent fields of a hypersurface*, or formally *derivations* in algebraic geometry. Finally we sketch some recent IBP methods [Ita15, LZ16].

4.1 Baikov representation

The basic idea of the Baikov representation [Bai96] is to define inverse propagators and irreducible scalar products (ISP), except μ_{ij} 's, as free variables.

For a simple example, consider the $D = 4 - 2\epsilon$ one-loop box diagram (2.38).

$$I_{\text{box}}^D[N] = \int \frac{d^D l}{i\pi^{D/2}} \frac{N}{D_1 D_2 D_3 D_4}. \quad (4.3)$$

Using van Neerven-Vermaseren variables, there are 5 variables x_1, x_2, x_3, x_4 and μ_{11} . Hence it is a 5-variable system. The solid angle of (-2ϵ) directions in this integral is irrelevant, hence,

$$\begin{aligned} I_{\text{box}}^D[N] &= \frac{1}{i\pi^{D/2}} \int d^{-2\epsilon} l^\perp \int d^4 l^{[4]} \frac{N}{D_1 D_2 D_3 D_4} \\ &= \frac{1}{i\pi^{D/2}} \frac{\pi^{\frac{D-4}{2}}}{\Gamma(\frac{D-4}{2})} \int_0^\infty \mu_{11}^{\frac{D-6}{2}} d\mu_{11} \int d^4 l^{[4]} \frac{N}{D_1 D_2 D_3 D_4} \end{aligned}$$

$$= \frac{1}{i\pi^2\Gamma(\frac{D-4}{2})} \frac{2}{t(t+s)} \int_0^\infty \mu_{11}^{\frac{D-6}{2}} d\mu_{11} \int dx_1 dx_2 dx_3 dx_4 \frac{N}{D_1 D_2 D_3 D_4}, \quad (4.4)$$

where the factor $2/(t(t+s))$ is the Jacobian of changing variables $l^{[4]} \rightarrow x_1, \dots, x_4$. Note that in this form, the dimension shift identities (2.46) are manifest.

Since in this case the ISPs are x_4 and μ_{11} , we define Baikov variables z_1, \dots, z_5 as,

$$z_1 \equiv D_1, \quad z_2 \equiv D_2, \quad z_3 \equiv D_3, \quad z_4 \equiv D_4, \quad z_5 \equiv l_1 \cdot \omega, \quad (4.5)$$

Note that the Jacobian

$$\frac{\partial(z_1, z_2, z_3, z_4, z_5)}{\partial(x_1, x_2, x_3, x_4, \mu_{11})} = -8 \quad (4.6)$$

is a constant. This is not surprising since by (2.42),

$$x_1 = \frac{1}{2}(z_1 - z_2), \quad x_2 = \frac{1}{2}(z_2 - z_3) + \frac{s}{2}, \quad x_3 = \frac{1}{2}(z_4 - z_1), \quad (4.7)$$

$z_5 = x_4$ and D_1 is linear in μ_{11} . The inverse map, $(z_1, z_2, z_3, z_4, z_5) \mapsto (x_1, x_2, x_3, x_4, \mu_{11})$ uniquely exists and has polynomial form,

$$\begin{aligned} \mu_{11} = \frac{1}{4st(s+t)} & (s^2 t^2 - 2s^2 t z_2 - 2s^2 t z_4 + s^2 z_2^2 + s^2 z_4^2 - 4s^2 z_5^2 - 2s^2 z_2 z_4 \\ & - 2st^2 z_1 - 2st^2 z_3 + 2stz_1 z_2 - 4stz_1 z_3 + 2stz_2 z_3 + 2stz_1 z_4 - 4stz_2 z_4 + 2stz_3 z_4 + t^2 z_1^2 \\ & + t^2 z_3^2 - 2t^2 z_1 z_3) \equiv F(z_1, z_2, z_3, z_4, z_5). \end{aligned} \quad (4.8)$$

Then, we get the Baikov representation,

$$I_{\text{box}}[N] = \frac{1}{i\pi^2\Gamma(\frac{D-4}{2})} \frac{1}{4t(t+s)} \int_{\Omega} dz_1 dz_2 dz_3 dz_4 dz_5 F(z_1, z_2, z_3, z_4, z_5)^{\frac{D-6}{2}} \frac{N}{z_1 z_2 z_3 z_4}, \quad (4.9)$$

where $F(z_1, z_2, z_3, z_4, z_5)$ is called the Baikov polynomial. N is a polynomial in z_1, \dots, z_5 . The integral region Ω is defined by $F(z_1, z_2, z_3, z_4, z_5) \geq 0$. In general, the integral region of Baikov representation is complicated. However, for the purpose of deriving IBPs, the explicit region is not important².

In practice, after OPP integrand reduction [OPP07, OPP08], N is a polynomial of μ_{11} and at most linear in $(l \cdot \omega)$ (2.44). The terms with μ_{11} lead to scalar integrals in higher dimension (2.46), while terms linear in $(l \cdot \omega)$ are spurious. Hence we assume that N is independent of $(l \cdot \omega)$ and μ_{11} . That implies that we can *integrate out* the ω direction.

Define $V = \text{span}\{k_1, k_2, k_4\}$ and V^\sharp is the direct sum of $\text{span}\{\omega\}$ and (-2ϵ) -dimensional spacetime. We decompose $l = l^{[3]} + l^\sharp$ according to $V \oplus V^\sharp$. Then

$$(l^\sharp)^2 = -\mu_{11} - \frac{s}{t(s+t)} x_4^2 \equiv -\lambda_{11}. \quad (4.10)$$

It is clear that D_1, \dots, D_4 are functions in x_1, x_2, x_3 and λ_{11} only. We may redefine the

²Note that the boundary of the integration is defined by the hypersurface $F = 0$, therefore the total derivatives vanish on the boundary and we do not need to worry about surface terms in IBPs.

Baikov variables,

$$z_1 = D_1, \quad z_2 = D_2, \quad z_3 = D_3, \quad z_4 = D_4. \quad (4.11)$$

Only 4 variables are needed. Repeat the previous process,

$$I_{\text{box}}[N] = \frac{1}{i\pi^{3/2}\Gamma(\frac{D-3}{2})} \frac{1}{4\sqrt{-st(t+s)}} \int dz_1 dz_2 dz_3 dz_4 \tilde{F}(z_1, z_2, z_3, z_4)^{\frac{D-5}{2}} \frac{N}{z_1 z_2 z_3 z_4}. \quad (4.12)$$

if N has no $l_1 \cdot \omega$ dependence. $\tilde{F}(z_1, z_2, z_3, z_4) = F(z_1, z_2, z_3, z_4, 0)$.

Baikov representation also works for higher-loop and both planar and nonplanar diagrams. For example, in a scheme within which all external particles are in $4D$, a two-loop integral with $n \geq 5$ points becomes

$$I_{n \geq 5}^{(2)}[N] = \frac{2^{D-6}}{\pi^5 \Gamma(D-5) J} \int \prod_{i=1}^{11} dz_i F(z)^{\frac{D-7}{2}} \frac{N}{z_1 \cdots z_k}, \quad (4.13)$$

where J is a Jacobian without D dependence. Here $F(z)$ is the determinant $\mu_{11}\mu_{22} - \mu_{12}^2$ in Baikov representation. In the same scheme, for a two-loop amplitude with $n < 5$ point, we can integrate out $5 - n$ spurious directions and get,

$$I_{n < 5}^{(2)}[N] = \frac{2^{D-n-1}}{\pi^n \Gamma(D-n) J} \int \prod_{i=1}^{2n+1} dz_i F(z)^{\frac{D-n-2}{2}} \frac{N}{z_1 \cdots z_k}. \quad (4.14)$$

We leave the Baikov representation of the massless double box diagram as an exercise (Exercise 4.1).

For deriving IBP relations, the overall prefactors are irrelevant. In the rest of this chapter, we neglect these factors in Baikov representation. In general for an L -loop integral in a scheme of which external particles are in $4D$,

$$I_n^{(L)}[N] \propto \int \prod_{i=1}^{\phi(n)L + \frac{L(L-1)}{2}} dz_i F(z)^{\frac{D-L-\phi(n)}{2}} \frac{N}{z_1 \cdots z_k}, \quad (4.15)$$

where

$$\phi(n) = \begin{cases} n, & n < 5 \\ 5, & n \geq 5 \end{cases}. \quad (4.16)$$

The Baikov polynomial $F(z)$ is the determinant $\det(\mu_{ij})$ if $n \geq 5$, or the determinant $\det(\lambda_{ij})$ is $n < 5$.

4.1.1 Unitarity cuts in Baikov representation

We see that in the Baikov representation, inverse propagators are simply linear monomials. Another feature is that the unitarity cut structure is clear.

Note that now all inverse propagators are linear, so a unitarity cut $D_i^{-1} \rightarrow \delta(D_i)$ just means to set certain z_i as zero in (4.15). For a given c -fold cut ($0 \leq c \leq k$), let \mathcal{S}_{cut} , $\mathcal{S}_{\text{uncut}}$ and \mathcal{S}_{ISP} be the sets of indices labelling cut propagators, uncut propagators and ISPs, respectively. \mathcal{S}_{cut} thus contains c elements. Furthermore, we denote m as the total

number of z_j variables,

$$m = \phi(n)L + \frac{L(L-1)}{2}, \quad (4.17)$$

and set $\mathcal{S}_{\text{uncut}} = \{r_1, \dots, r_{k-c}\}$ and $\mathcal{S}_{\text{ISP}} = \{r_{k-c+1}, \dots, r_{m-c}\}$. Then, by cutting the propagators, $z_i^{-1} \rightarrow \delta(z_i)$, $i \in \mathcal{S}_{\text{cut}}$, the integrals (4.13) and (4.14) reduce to

$$I_{\text{cut}}^{(L)}[N] = \int \frac{dz_{r_1} \cdots dz_{r_{m-c}}}{z_{r_1} \cdots z_{r_{k-c}}} NF(z)^{\frac{D-L-\phi(n)}{2}} \Big|_{z_i=0, \forall i \in \mathcal{S}_{\text{cut}}}, \quad (4.18)$$

Example 4.1. Consider the quintuple cut for the D -dimensional massless double box. (See Exercise 4.1), $D_2 = D_3 = D_5 = D_6 = D_7 = 0$. In this cases, $m = 9$. $\mathcal{S}_{\text{uncut}} = \{1, 4\}$, $\mathcal{S}_{\text{cut}} = \{2, 3, 5, 6, 7\}$, $\mathcal{S}_{\text{ISP}} = \{8, 9\}$. The Baikov representation (4.18) with this cut reads,

$$I_{\text{penta-cut}}^{(2)}[N] = \int \frac{dz_1 dz_4 dz_8 dz_9}{z_1 z_4} F_{[5]}(z)^{\frac{D-6}{2}} N \Big|_{z_2=z_3=z_5=z_6=z_7=0}, \quad (4.19)$$

where,

$$F_{[5]}(z) = \frac{(st - sz_1 - 2sz_8 - 2sz_9 - tz_1 - tz_4 + 2z_4 z_8 - 4z_8 z_9)(2sz_1 z_9 + 4sz_8 z_9 + tz_1 z_4)}{4st(s+t)}. \quad (4.20)$$

If we consider the maximal cut $D_1 = D_2 = \dots = D_7 = 0$, then $\mathcal{S}_{\text{uncut}} = \emptyset$, $\mathcal{S}_{\text{cut}} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{S}_{\text{ISP}} = \{8, 9\}$. The Baikov representation (4.18) on this cut reads,

$$I_{\text{hepta-cut}}^{(2)}[N] = \int dz_8 dz_9 F_{[7]}(z)^{\frac{D-6}{2}} N \Big|_{z_i=0, 1 \leq i \leq 7}, \quad (4.21)$$

and the Baikov polynomial on the maximal cut is simply,

$$F_{[7]}(z) = \frac{z_8 z_9 (st - 2sz_8 - 2sz_9 - 4z_8 z_9)}{t(s+t)}. \quad (4.22)$$

4.1.2 IBPs in Baikov representation

Note that the higher the unitarity cut is, the simpler the Baikov polynomial becomes. So we try to use cuts as much as possible to reconstruct the full IBP, instead of finding the full set of IBPs at once. Suppose that we consider a c -fold cut and make an IBP ansatz as,

$$\begin{aligned} 0 &= \int d \left(\sum_{i=1}^{m-c} \frac{(-1)^{i+1} a_{r_i} F(z)^{\frac{D-h}{2}}}{z_{r_1} \cdots z_{r_{k-c}}} dz_{r_1} \wedge \cdots \widehat{dz_{r_i}} \cdots \wedge dz_{r_{m-c}} \right) \\ &= \int \sum_{i=1}^{m-c} \left(\frac{\partial a_{r_i}}{\partial z_{r_i}} \right) F(z)^{\frac{D-h}{2}} \omega + \frac{D-h}{2} \sum_{i=1}^{m-c} \left(a_{r_i} \frac{\partial F}{\partial z_{r_i}} \right) F^{\frac{D-h-2}{2}} \omega - \sum_{i=1}^{k-c} \frac{a_{r_i}}{z_{r_i}} F(z)^{\frac{D-h}{2}} \omega. \end{aligned} \quad (4.23)$$

where ω is the measure $dz_{r_1} \wedge \dots \wedge dz_{r_{m-c}} / (z_{r_1} \cdots z_{r_{k-c}})$ and $h = D - L - \phi(n)$. The second sum contains integrals in $D - 2$ dimension while the third sum contains doubled propagators.

If it is required that resulting IBP has no dimensional shift or doubled poles [Ita15, LZ16], we have the *syzygy equations*,

$$bF + \sum_{i=1}^{m-c} a_{r_i} \frac{\partial F}{\partial z_{r_i}} = 0, \quad (4.24)$$

$$a_{r_i} + b_{r_i} z_{r_i} = 0, \quad i = 1, \dots, k - c, \quad (4.25)$$

where a_{r_i} , b and b_{r_i} must be polynomials in z_j . Note that the last $(k - c)$ equations in Eq. (4.25) are trivial since they are solved as $a_{r_i} = -b_{r_i} z_{r_i}$. So alternatively, we have only one syzygy equation,

$$bF - \sum_{i=1}^{k-c} b_{r_i} \left(z_{r_i} \frac{\partial F}{\partial z_{r_i}} \right) + \sum_{j=k-c+1}^{m-c} a_{r_j} \frac{\partial F}{\partial z_{r_j}} = 0 \quad (4.26)$$

for polynomials b_{r_i} , a_{r_i} and b . These equations are similar to the tangent condition of a hypersurface in differential geometry, however, we require polynomial solutions. So we apply algebraic geometry to study these equations.

4.2 Syzygies

Syzygy can be understood as relations between polynomials. Consider the ring $R = \mathbb{F}[z_1, \dots, z_n]$ and R^m , the set of all m -tuple of R . R^m in general is not a ring but $R \times R^m \rightarrow R$ is well-defined as,

$$f \cdot (f_1, \dots, f_m) \mapsto (ff_1, \dots, ff_m). \quad (4.27)$$

This leads to the definition of modules.

Definition 4.2. A module M over a ring R is an Abelian group $(+)$ with a map $R \times M \rightarrow M$ such that,

1. $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2, \forall r \in R, m_1, m_2 \in M.$
2. $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m, \forall r_1, r_2 \in R, m \in M.$
3. $(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m), \forall r_1, r_2 \in R, m \in M.$
4. $1 \cdot m = m. 1 \in R, \forall m \in M.$

For example R^m , I and R/I are all R -modules, where I is an ideal of R . To simplify notations, we formally write an element $(f_1, \dots, f_m) \in R^m$ as $f_1 \mathbf{e}_1 + \dots, f_m \mathbf{e}_m$.

Proposition 4.3. Any submodule of R^m is finitely generated.

Proof. This is a generalization of Theorem 2.3. See Cox, Little and O'Shea [CLO98]. \square

Definition 4.4. Given an R module M , the syzygy module of $m_1, \dots, m_k \in M$, $\text{syz}(m_1 \dots m_k)$, is the submodule of R^k which consists of all (a_1, \dots, a_k) such that

$$a_1 \cdot m_1 + a_2 \cdot m_2 + \dots + a_k \cdot m_k = 0. \quad (4.28)$$

So (4.26) defines a syzygy module with $M = R$, i.e., “relations” between polynomials. Naively, given f_1, \dots, f_k , it is clear that $f_j \mathbf{e}_i - f_i \mathbf{e}_j \in R^k$, $i \neq j$ is a syzygy for f_1, \dots, f_k . Such a syzygy is called a *principal syzygy* which is denoted as P_{ij} .

In some cases, principal syzygies generate the whole syzygy module of given polynomials. For example,

Proposition 4.5. Given f_1, \dots, f_k in $R = \mathbb{F}[z_1, \dots, z_n]$, if $\langle f_1, \dots, f_k \rangle = \langle 1 \rangle$, then $\text{syz}(f_1, \dots, f_k)$ is generated by the principal syzygies $P_{ij} = f_j \mathbf{e}_i - f_i \mathbf{e}_j$, $1 \leq i \neq j \leq k$.

Proof. We have $q_1 f_1 + \dots + q_k f_k = 1$, where the q_i 's are in R . For any element in $\text{syz}(f_1, \dots, f_k)$,

$$a_1 f_1 + \dots + a_k f_k = 0. \quad (4.29)$$

we can rewrite a_i as,

$$a_i = \sum_{j=1}^k a_i q_j f_j = \left(\sum_{\substack{j=1 \\ j \neq i}}^k a_i q_j f_j \right) + a_i q_i f_i = \sum_{\substack{j=1 \\ j \neq i}}^k a_i q_j f_j - \sum_{\substack{j=1 \\ j \neq i}}^k a_j q_i f_j \equiv \sum_{\substack{j=1 \\ j \neq i}}^k s_{ij} f_j,$$

where $s_{ij} = a_i q_j - a_j q_i$ is a polynomial and is antisymmetric in indices. Hence, this syzygy

$$\sum_{i=1}^k a_i \mathbf{e}_i = \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k s_{ij} f_j \mathbf{e}_i = \sum_{\substack{i=1, j=1 \\ j \neq i}}^k \frac{s_{ij}}{2} (f_j \mathbf{e}_i - f_i \mathbf{e}_j) = \sum_{\substack{i=1, j=1 \\ j \neq i}}^k \frac{s_{ij}}{2} P_{ij}, \quad (4.30)$$

is generated by principal syzygies. \square

Example 4.6. Consider the polynomial $F = x^2 + y^2 - 1$ in $\mathbb{Q}[x, y]$. Define $f_1 = \partial F / \partial x = 2x$, $f_2 = \partial F / \partial y = 2y$ and $f_3 = F$. It is clear that $\langle 2x, 2y, x^2 + y^2 - 1 \rangle = \langle 1 \rangle$. Hence $\text{syz}(f_1, f_2, f_3)$ is generated by,

$$(y, -x, 0), \quad (x^2 + y^2 - 1, 0, -2x), \quad (0, x^2 + y^2 - 1, -2y). \quad (4.31)$$

Note that we see that $F = 0$ defines the unit circle. The tangent vector at any point on the circle is,

$$y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad (4.32)$$

which corresponds to the first generator in (4.6).

In general, syzygy module for given polynomials can be found by Gröbner basis computation. For a Gröbner basis $G = \{g_1, \dots, g_m\}$ in a certain monomial order, consider two elements g_i, g_j , $i < j$. Let $S(g_i, g_j) = a_i g_i + a_j g_j$ be the S-polynomial (Definition 2.17).

$S(g_i, g_j)$ must be divisible by G , hence, by polynomial division (Algorithm 2),

$$a_i g_i + a_j g_j = \sum_{l=1}^m q_l g_l. \quad (4.33)$$

Clearly, this is a syzygy of g_1, \dots, g_m , which explicitly reads $q_1 \mathbf{e}_1 + \dots (q_i - a_i) \mathbf{e}_i + \dots + (q_j - a_j) \mathbf{e}_j + \dots q_m \mathbf{e}_m$. We call this syzygy, *reduction of an S -polynomial* and denote it as $s_{ij} \equiv \sum_{l=1}^m (s_{ij})_l \mathbf{e}_l$.

Theorem 4.7 (Schreyer).

1. For a Gröbner basis $G = \{g_1, \dots, g_m\}$ in $R = \mathbb{F}[z_1, \dots, z_n]$, $\text{syz}(g_1, \dots, g_m)$ is generated by reductions of S -polynomials, s_{ij} .
2. For generic polynomials $\{f_1, \dots, f_k\}$ in $R = \mathbb{F}[z_1, \dots, z_n]$, let $G = \{g_1, \dots, g_m\}$ be their Gröbner basis in a certain monomial order. Suppose that the conversion relations are,

$$g_i = \sum_{j=1}^k a_{ij} f_j \quad f_i = \sum_{j=1}^m b_{ij} g_j. \quad (4.34)$$

The $\text{syz}(f_1, \dots, f_k)$ is generated by,

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^k (s_{\alpha\beta})_i a_{ij} \mathbf{e}_j, \quad 1 \leq \alpha < \beta \leq m \\ \mathbf{e}_i - \sum_{l=1}^k \sum_{j=1}^m b_{ij} a_{jl} \mathbf{e}_l, \quad 1 \leq i \leq k \end{aligned} \quad (4.35)$$

Proof. See Cox, Little and O'Shea [CLO98]. Note that in second line of (4.35), the relations are coming from the map from f_i 's to G and the inverse map. \square

This theorem also generalizes to modules. Given several elements m_1, \dots, m_k in R^m , we can define a module order which is an extension of monomial order. Then we can compute Gröbner basis and the syzygy module of m_1, \dots, m_k [CLO98].

In practice, we may use **syz** in SINGULAR or **syz** in MACAULAY2, to find the syzygy module of polynomials or elements in R^m . See alternative ways of finding syzygies with the linear algebra method [Sch12] or by F5 algorithm [AH11].

4.3 Polynomial tangent vector field

In this section, we use the tool of syzygy to study *polynomial tangent vector field* [HM93].

Let $F(z)$ be a polynomial in $R = \mathbb{C}[z_1, \dots, z_n]$. $F = 0$ defines a hypersurface (reducible or irreducible). The set of polynomial tangent fields, \mathbf{T}_F , is the submodule in R^n which consists of all $(a_1, \dots, a_n) \in R^n$ such that,

$$\sum_{i=1}^n a_i \frac{\partial F}{\partial z_i} = bF, \quad (4.36)$$

for some polynomial b in z_i 's. (4.36) is a syzygy equation which can be solved by the algorithm in Theorem 4.7. (We drop the factor b in the definition of \mathbf{T}_F , since this factor can be easily recovered later.) Mathematically, \mathbf{T}_F is called the set of *derivations*, from $R/\langle F \rangle$ to $R/\langle F \rangle$.

Geometrically, if a point $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ is on the hypersurface $\mathcal{Z}(F)$, then

$$\sum_{i=1}^n a_i(\xi_1, \dots, \xi_n) \frac{\partial F}{\partial z_i}(\xi_1, \dots, \xi_n) = 0, \quad (4.37)$$

and $(a_1(\xi), \dots, a_n(\xi))$ is along the tangent direction of $\mathcal{Z}(F)$. This is the origin of the terminology, *polynomial tangent vector field*.

Although the syzygy computation by Theorem 4.7 can find \mathbf{T}_F for any polynomial F , it is interesting to study the geometric properties of F and \mathbf{T}_F .

Definition 4.8. For a polynomial F in $R = \mathbb{C}[z_1, \dots, z_n]$, the singular ideal I_s for F is defined to be,

$$I_s = \left\langle \frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n}, F \right\rangle. \quad (4.38)$$

If $I_s = \langle 1 \rangle$, then we call the hypersurface $\mathcal{Z}(F)$ smooth. Otherwise we call points in $\mathcal{Z}(I_s)$ singular points.

Intuitively, at a singular point $\xi \in \mathcal{Z}(I_s)$, F and all its first derivatives vanish. Hence near ξ , $F = 0$ does not define a complex submanifold with codimension 1.

If a hypersurface is smooth, then by Definition 4.8 and Proposition 4.5, we have the following statement.

Proposition 4.9. If F in $R = \mathbb{C}[z_1, \dots, z_n]$ defines a smooth hypersurface, then \mathbf{T}_F is generated by principal syzygies of $\partial F / \partial z_1, \dots, \partial F / \partial z_n, F$.

For instance, in Example 4.6, the unit circle is clearly smooth. Hence its polynomial tangent vector fields is generated by principal syzygies. This can be understood as an algebraic version of the implicit function theorem.

The singular cases are more interesting and subtle.

Example 4.10. Let $F = y^2 - x^3$. $F = 0$ is not a smooth curve, since the singular variety is $I_s = \langle -3x^2, 2y, y^2 - x^3 \rangle = \langle x^2, y \rangle \neq \langle 1 \rangle$. So there is one singular point at $(0, 0)$ which is a cusp point. We cannot just use principal syzygies to generate \mathbf{T}_F , so we turn to Theorem 4.7.

Define $\{f_1, f_2, f_3\} = \{-3x^2, 2y, y^2 - x^3\}$. Note that this is a Gröbner basis in grevlex, although it is not a reduced Gröbner basis.

- $S(f_1, f_2) = (2y)f_1 + (3x^2)f_2 = 0$ hence we get a syzygy generator $\mathcal{S}_1 = (2y, -3x^2, 0)$.
- $S(f_2, f_3) = (-x^3)f_2 - (2y)f_3 = -2y^3 = -y^2f_2$. $\mathcal{S}_2 = (0, -x^3 + y^2, -2y)$.
- $S(f_3, f_1) = -3f_3 + xf_1 = -3y^2 = -\frac{3}{2}yf_2$. $\mathcal{S}_3 = (x, \frac{3}{2}y, -3)$.

\mathcal{S}_3 is not from principal syzygies. Locally it characterizes the scaling behavior of the curve $y^2 - x^3 = 0$ near the cusp point $(0, 0)$. It is a weighted Euler vector field [HM93].

Dropping the factor b in (4.36), we find that \mathbf{T}_F is generated by,

$$(2y, -3x^2), \quad (0, -x^3 + y^2), \quad \left(x, \frac{3y}{2}\right). \quad (4.39)$$

Proposition 4.11. Let $F \in R = \mathbb{C}[z_1, \dots, z_n]$, \mathbf{T}_F is a Lie algebra with $[\cdot, \cdot]$ defined as that for vector fields.

Proof. Let $v_1 = (a_1, \dots, a_n)$ and $v_2 = (b_1, \dots, b_n)$ be two polynomial tangent vector fields,

$$\sum_{i=1}^n a_i \frac{\partial F}{\partial z_i} = AF, \quad \sum_{i=1}^n b_i \frac{\partial F}{\partial z_i} = BF, \quad (4.40)$$

where A and B are polynomials. $[v_1, v_2]$'s i -th component is,

$$\sum_{j=1}^n \left(a_j \frac{\partial b_i}{\partial z_j} - b_j \frac{\partial a_i}{\partial z_j} \right), \quad (4.41)$$

Hence $[v_1, v_2]$ acts on F as,

$$\sum_{i=1}^n \sum_{j=1}^n \left(a_j \frac{\partial b_i}{\partial z_j} - b_j \frac{\partial a_i}{\partial z_j} \right) \frac{\partial F}{\partial z_i} = F \cdot \sum_{j=1}^n \left(a_j \frac{\partial B}{\partial z_j} - b_j \frac{\partial A}{\partial z_j} \right). \quad (4.42)$$

so $[v_1, v_2]$ is in \mathbf{T}_F . □

In general, \mathbf{T}_F is an infinite-dimensional Lie algebra over \mathbb{C} . We may call \mathbf{T}_F a *tangent algebra*.

If we require a polynomial vector field (a_1, \dots, a_n) to be tangent to a list of hypersurfaces defined by F_1, \dots, F_k , like the case of (4.24) and (4.25),

$$\begin{aligned} \sum_{i=1}^n a_i \frac{\partial F_1}{\partial z_i} &= A_1 F_1(z) \\ &\dots \\ \sum_{i=1}^n a_i \frac{\partial F_k}{\partial z_i} &= A_k F_k(z). \end{aligned} \quad (4.43)$$

Then by definition, the solution set of such (a_1, \dots, a_n) 's is the intersection of modules $\mathbf{T}_{F_1} \cap \dots \cap \mathbf{T}_{F_k}$, which is again a submodule of R^n . On the other hand,

Proposition 4.12. If a polynomial F in $R = \mathbb{C}[z_1, \dots, z_n]$ factorizes as,

$$F = f_1^{s_1} \dots f_k^{s_k}, \quad (4.44)$$

where f_i 's are irreducible polynomials in R and $f_i \nmid f_j$ if $i \neq j$. Here s_i 's are positive integers. Then $\mathbf{T}_F = \mathbf{T}_{f_1} \cap \dots \cap \mathbf{T}_{f_k}$.

Proof. It is clear that $\mathbf{T}_F \supset \mathbf{T}_{f_1} \cap \dots \cap \mathbf{T}_{f_k}$. For $(a_1, \dots, a_n) \in \mathbf{T}_F$,

$$\sum_{l=1}^k s_l \left(\sum_{i=1}^n a_i \frac{\partial f_l}{\partial z_i} \right) \frac{F}{f_l} = bF. \quad (4.45)$$

For a fixed index t , $1 \leq t \leq k$, divide the above expression by $f_t^{s_t-1}$,

$$s_t \left(\sum_{i=1}^n a_i \frac{\partial f_t}{\partial z_i} \right) \frac{F}{f_t^{s_t}} + \sum_{\substack{l=1 \\ l \neq t}}^k s_l \left(\sum_{i=1}^n a_i \frac{\partial f_l}{\partial z_i} \right) \frac{F}{f_l f_t^{s_t-1}} = b \frac{F}{f_t^{s_t-1}}. \quad (4.46)$$

Note that the second term on the l.h.s and the r.h.s are polynomials proportional to f_t . Hence,

$$s_t \left(\sum_{i=1}^n a_i \frac{\partial f_t}{\partial z_i} \right) \frac{F}{f_t^{s_t}}, \quad (4.47)$$

is also proportional to f_t . However f_t does not divide $F/f_t^{s_t}$, since f_i 's are distinct irreducible polynomials. So f_t divides $\sum_{i=1}^n a_i \partial f_t / \partial z_i$ and $(a_1, \dots, a_n) \in \mathbf{T}_{f_t}$, and $\mathbf{T}_F \subset \mathbf{T}_{f_1} \cap \dots \cap \mathbf{T}_{f_k}$. \square

It implies that for a reducible hypersurface, its tangent algebra is the intersection of tangent algebras of all its irreducible components [HM93].

In practice, give a syzygy equation system (4.43), we can first determine each \mathbf{T}_{F_i} and then calculate the intersection $\mathbf{T}_{F_1} \cap \dots \cap \mathbf{T}_{F_k}$. (See [CLO98, Chapter 5] for the algorithm of computing intersection of submodules.) Furthermore, for each \mathbf{T}_{F_i} , if F_i is factorable, we can use Proposition 4.12 to further divide the problem. This divide-and-conquer approach is in general much more efficient than solving (4.43) at once.

4.4 IBPs from syzygies and unitarity

With the Baikov representation, unitarity cut and syzygy computations, we introduce some recent IBP generating algorithms [Ita15, LZ16] with the two-loop double box as an example.

For the massless double box, define

$$I[m_1, \dots, m_9] = \int \frac{d^D l_1}{i\pi^{D/2}} \frac{d^D l_2}{i\pi^{D/2}} \frac{(l_1 \cdot k_4)^{-m_8} (l_2 \cdot k_1)^{-m_9}}{D_1^{m_1} \dots D_7^{m_7}}. \quad (4.48)$$

Our target integral space is the set of all list $(m_1 \dots m_9)$ such that $m_i \leq 1$, $i = 1, \dots, 7$, $m_j \leq 0$, $j = 8, \dots, 9$, since we try to find IBPs without doubled propagators.

Example 4.13. Consider the massless double box with maximal cut. From Example 4.1, we see that with maximal cut the Baikov polynomial is

$$F_{[7]} = \frac{z_8 z_9 (st - 2sz_8 - 2sz_9 - 4z_8 z_9)}{t(s+t)}. \quad (4.49)$$

The syzygy equation is,

$$a_8 \frac{\partial F_{[7]}}{\partial z_8} + a_9 \frac{\partial F_{[7]}}{\partial z_9} = \beta F_{[7]}. \quad (4.50)$$

Solutions of (a_8, a_9) form $\mathbf{T}_{F_{[7]}}$, the tangent algebra of $F_{[7]}$. We leave the computation of $\mathbf{T}_{F_{[7]}}$ as an exercise. There are 3 generators of $\mathbf{T}_{F_{[7]}}$,

$$\begin{aligned} v_1 &= (-(t - 2z_8)z_8, (t - 2z_9)z_9), & v_2 &= (2(s + t)z_8z_9, -(t - 2z_9)z_9(s + 2z_9)), \\ v_3 &= (0, -z_9(st - 2sz_8 - 2sz_9 - 4z_8z_9)). \end{aligned} \quad (4.51)$$

Using these generators and the ansatz (4.23), we get IBPs without double propagators. For instance, from the first generator we have the IBP,

$$I[1, 1, 1, 1, 1, 1, 1, -1, 0] = I[1, 1, 1, 1, 1, 1, 1, 0, -1] + \dots, \quad (4.52)$$

and from the second generator,

$$\begin{aligned} &4(D - 3)I[1, 1, 1, 1, 1, 1, 1, 0, -2] + (3Ds - 12s - 2t)I[1, 1, 1, 1, 1, 1, 1, 0, -1] \\ &\quad - \frac{1}{2}(D - 4)stI[1, 1, 1, 1, 1, 1, 1, 0, 0] = 0 + \dots \end{aligned} \quad (4.53)$$

Note that with maximal cut, any integral with at least one $m_i < 1$, $i = 1, \dots, 7$, is neglected. “...” stands for these integrals.

To get all IBPs with maximal cut, we need to consider vector fields $q_1v_1 + q_2v_2 + q_3v_3$ where q_1 , q_2 and q_3 are arbitrary polynomials in z_8 , z_9 up to a given degree. When the smoke is clear, we find that all integrals with $m_i = 1$, $i = 1, \dots, 7$ and $m_j \leq 0$, $j = 8, \dots, 9$ are reduced to $I[1, 1, 1, 1, 1, 1, 1, 0, 0]$, $I[1, 1, 1, 1, 1, 1, 1, -1, 0]$ and integrals with fewer-than-7 propagators.

Example 4.14. Consider the quintuple cut of the massless double box, $D_2 = D_3 = D_5 = D_6 = D_7 = 0$. The goal is to study integrals $I_{dbox}[m_1, m_2, \dots, m_9]$ such that $m_2 = m_3 = m_5 = m_6 = m_7 = 1$, $m_1, m_4 \leq 1$, m_8, m_9 non-positive. The syzygy equations read,

$$a_1 \frac{\partial F_{[5]}}{\partial z_1} + a_4 \frac{\partial F_{[5]}}{\partial z_4} + a_8 \frac{\partial F_{[5]}}{\partial z_8} + a_9 \frac{\partial F_{[5]}}{\partial z_9} = \beta F_{[5]} \quad (4.54)$$

$$a_1 = b_1 z_1 \quad (4.55)$$

$$a_4 = b_4 z_4 \quad (4.56)$$

In the formal language, the solutions of the last two equations form a tangent algebra \mathbf{T}_{14} with generators,

$$(z_1, 0, 0, 0), \quad (0, z_4, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1). \quad (4.57)$$

The first equation can be solved by **syz** in SINGULAR and MACAULAY2, which leads to a tangent algebra $\mathbf{T}_{F_{[5]}}$. Then the solution set of (4.56) of $\mathbf{T}_{F_{[5]}} \cap \mathbf{T}_{14}$. This intersection of submodules can be calculated by **intersect** in SINGULAR and MACAULAY2.

Again we find IBPs with this tangent algebra. All integrals with $m_2 = m_3 = m_5 = m_6 = m_7 = 1$, $m_1, m_4 \leq 1$, m_8, m_9 non-negative are reduced to 3 master integrals

$I[1, 1, 1, 1, 1, 1, 1, 0, 0]$, $I[1, 1, 1, 1, 1, 1, 1, -1, 0]$, $I[0, 1, 1, 0, 1, 1, 1, 0, 0]$ and integrals with fewer-than-5 propagators.

In general it is easy to obtain IBPs with maximal cut, since the number of variable is small. We may use symmetries and IBPs with maximal cut, numerically, to find all MIs in our package AZURITE [GLZ16]. (Alternatively, the master integral list can be obtained by the critical point analysis in parametric representation [LP13].) It takes only a few seconds to find all master integrals for the massless double box.

- double box, $I[1, 1, 1, 1, 1, 1, 1, -1, 0]$, $I[1, 1, 1, 1, 1, 1, 1, 0, 0]$,
- slashed box, $I[0, 1, 1, 0, 1, 1, 1, 0, 0]$,
- box bubble, $I[0, 1, 0, 1, 1, 1, 1, 0, 0]$,
- double bubble, $I[1, 0, 1, 1, 0, 1, 0, 0, 0]$,
- bubble triangle, $I[0, 1, 0, 1, 0, 1, 1, 0, 0]$,
- t -channel sunset, $I[0, 1, 0, 0, 1, 0, 1, 0, 0]$,
- s -channel sunset, $I[0, 0, 1, 0, 0, 1, 1, 0, 0]$.

We define that $I[m_1, \dots, m_9]$ is lower than $I[n_1, \dots, n_9]$ if $m_i \leq n_i$, $i = 1, \dots, 9$. For example, the s -channel sunset is lower than the slashed box. A triple cut $D_3 = D_6 = D_7 = 0$ contains all information of the quintuple cut in Example 4.14. Since here the lowest master integrals are double bubble, bubble triangle, t -channel sunset, s -channel sunset, we can see that the following four cuts,

$$\begin{aligned} D_1 = D_3 = D_4 = D_5 = 0, \quad D_2 = D_4 = D_6 = D_7 = 0 \\ D_2 = D_5 = D_7 = 0, \quad D_3 = D_6 = D_7 = 0, \end{aligned} \quad (4.58)$$

determine the complete IBPs without cut.

By this method [LZ16], a Mathematica code with communication with SINGULAR, analytically reduces all double box integrals with numerator rank ≤ 4 , to the 8 master integrals in about 39 seconds for the massless double box on a laptop. Similarly, it takes about 162 seconds for the analytic IBP reduction of the one-massive double box. We expect that combined with sparse linear algebra and finite-field fitting techniques, it can solve some very difficult two-loop/three-loop IBP problems in the near future.

4.5 Exercise

Exercise 4.1 (Baikov representation of massless double box). *Consider the two-loop massless double box diagram (Fig. 2.5) with inverse propagators D_1, \dots, D_7 defined in (2.3). Let*

$$I_{dbox}[N] = \int \frac{d^D l_1}{i\pi^{D/2}} \frac{d^D l_2}{i\pi^{D/2}} \frac{N}{D_1 \dots D_7}. \quad (4.59)$$

By integrand reduction, we see that N can be a polynomial in μ_{11} , μ_{22} and μ_{12} , but at most linear in $(l_1 \cdot \omega)$ and $(l_2 \cdot \omega)$. Terms linear in $(l_1 \cdot \omega)$ and $(l_2 \cdot \omega)$ are spurious so dropped. Terms in μ 's can be converted to integrals without μ 's in higher dimension, via Schwinger parameterization [BDFD03]. Or alternatively, polynomials in μ 's or $(l_i \cdot \omega)$ can be directly integrated out by adaptive integrand decomposition [MPP16], using Gegenbauer polynomials techniques. Hence we assume N contains no μ 's, $(l_1 \cdot \omega)$ or $(l_2 \cdot \omega)$.

1. The original Van Neerven-Vermaseren variables are defined in (2.54) and $\mu_{ij} = -l_i^\perp \cdot l_j^\perp$. To integrate out the ω direction, define $V_1 = \text{span}\{k_1, k_2, k_4\}$ and V^\sharp as the direct sum of $\text{span}\{\omega\}$ and (-2ϵ) extra spacetime. Decompose $l_i = l_i^{[3]} + l_i^\sharp$ and denote $(l_i^\sharp \cdot l_j^\sharp) = -\lambda_{ij}$. Prove that

$$\lambda_{11} = \mu_{11} + \frac{s}{t(s+t)}x_4^2, \quad \lambda_{22} = \mu_{22} + \frac{s}{t(s+t)}y_4^2, \quad \lambda_{12} = \mu_{12} + \frac{s}{t(s+t)}x_4y_4, \quad (4.60)$$

and the D_1, \dots, D_7 only depend on $x_1, x_2, x_3, y_1, y_2, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12}$.

2. Integrate over the solid angle parts of l_1^\sharp and l_2^\sharp to get

$$I_{dbox}[N] = \frac{2^{D-5}}{\pi^4 \Gamma(D-4)} \int_0^\infty d\lambda_{11} \int_0^\infty d\lambda_{22} \int_{-\sqrt{\lambda_{11}\lambda_{22}}}^{\sqrt{\lambda_{11}\lambda_{22}}} d\lambda_{12} (\lambda_{11}\lambda_{22} - \lambda_{12}^2)^{\frac{D-6}{2}} \times \int d^3l_1^{[3]} d^3l_2^{[3]} \frac{N}{D_1 \dots D_7}. \quad (4.61)$$

3. Define 9 Baikov variables as

$$z_i = D_i, \quad 1 \leq i \leq 7, \quad z_8 = l_1 \cdot k_4, \quad z_9 = l_2 \cdot k_1. \quad (4.62)$$

Find the inverse map $(z_1, \dots, z_9) \mapsto (x_1, x_2, x_3, y_1, y_2, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12})$ and the Jacobian of the map.

4. Derive the Baikov form of integral,

$$I_{dbox}[N] = \frac{2^{D-5}}{\pi^4 \Gamma(D-4)J} \int \prod_{i=1}^9 dz_i F(z)^{\frac{D-6}{2}} \frac{N}{D_1 \dots D_7}. \quad (4.63)$$

Calculate J and $F(z)$ explicitly. Note that the Jacobian of the changing variables $l_i^{[3]}$ to $(x_1, x_2, x_3, y_1, y_2, y_3)$ should be included.

Exercise 4.2. Derive the Baikov representation for two-loop pentagon-box diagram, (Fig. 4.1). with inverse propagators,

$$\begin{aligned} D_1 &= l_1^2, & D_2 &= (l_1 - k_1)^2, & D_3 &= (l_1 - k_1 - k_2)^2, & D_4 &= (l_1 - k_1 - k_2 - k_3)^2, \\ D_5 &= (l_2 + k_1 + k_2 + k_3)^2, & D_6 &= (l_2 + k_1 + k_2 + k_3 + k_4)^2, & D_7 &= l_2^2, & D_8 &= (l_1 + l_2)^2. \end{aligned} \quad (4.64)$$

(Hint: define $z_i = D_i$, $i = 1, \dots, 8$. $z_9 = l_1 \cdot k_5$, $z_{10} = l_2 \cdot k_1$, $z_{11} = l_2 \cdot k_2$.)

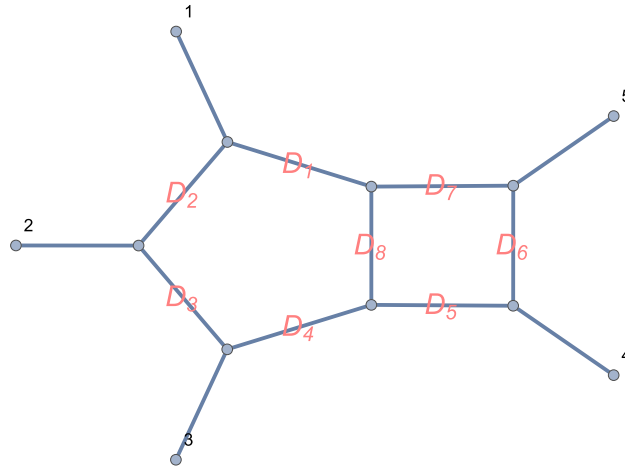


Figure 4.1: Pentagon box diagram

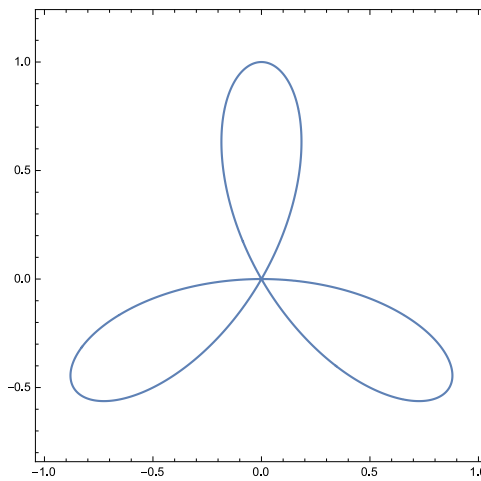
Exercise 4.3. Consider $f_1 = x^3 - 2xy$, $f_2 = x^2y - 2y^2 + x$ as the Example 2.21. We know that the Gröbner basis in grevlex is $G = \{g_1, g_2, g_3\} = \{x^2, xy, y^2 - \frac{1}{2}x\}$. The conversion relations are,

$$g_1 = -yf_1 + xf_2, \quad g_2 = -\frac{(1+xy)}{2}f_1 + \frac{1}{2}x^2f_2, \quad g_3 = -\frac{1}{2}y^2f_1 + \frac{1}{2}(xy-1)f_2, \quad (4.65)$$

$$f_1 = xg_1 - 2g_2, \quad f_2 = yg_1 - 2g_3. \quad (4.66)$$

Find the generators of $\text{syz}(f_1, f_2)$ by Theorem 4.7.

Exercise 4.4. Let $F = (x^2 + y^2)^2 + 3x^2y - y^3$, the plot of the curve $F = 0$ is in Figure 4.2. Determine the singular points of this curve and find the polynomial tangent vector fields \mathbf{T}_F .

Figure 4.2: A singular curve, $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$

Exercise 4.5. Compute \mathbf{T}_F for the double box on the maximal cut, where F is the corresponding Baikov polynomial (4.49). (We drop the subscript “[7]”.)

1. Use **syz** in SINGULAR or MACAULAY2, to compute \mathbf{T}_F directly.
2. Note that F has 3 irreducible factors, $f_1 = z_8$, $f_2 = z_9$ and $f_3 = (st - 2sz_8 - 2sz_9 - 4z_8z_9)$. f_1 is linear so \mathbf{T}_{f_1} is generated by,

$$(z_8, 0), \quad (0, 1). \quad (4.67)$$

Similarly, \mathbf{T}_{f_2} is generated by,

$$(1, 0), \quad (0, z_9). \quad (4.68)$$

What is $\mathbf{T}_{f_1} \cap \mathbf{T}_{f_2}$? Note that $f_3 = 0$ is smooth. Use Proposition 4.5 to find \mathbf{T}_{f_3} .

3. Use **intersection** in SINGULAR or MACAULAY2, to compute $\mathbf{T}_F = \mathbf{T}_{f_1} \cap \mathbf{T}_{f_2} \cap \mathbf{T}_{f_3}$. Compare the result with that from the direct computation.

Exercise 4.6. Consider the three-loop massless triple box diagram (Figure. 3.1).

1. Define $z_i = D_i$, $i = 1, \dots, 10$, and

$$\begin{aligned} z_{11} &= (l_1 + k_4)^2, & z_{12} &= (l_2 + k_1)^2, & z_{13} &= (l_3 + k_1)^2, \\ z_{14} &= (l_3 + k_4)^2, & l_{15} &= (l_1 + l_2)^2. \end{aligned} \quad (4.69)$$

Determine its Baikov representation.

2. Derive IBPs with the maximal cut $D_1 = \dots = D_{10} = 0$, and determine the master integrals with 10 propagators for this diagram.

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